# Stationary ring solitons in field theory – knots and vortons’

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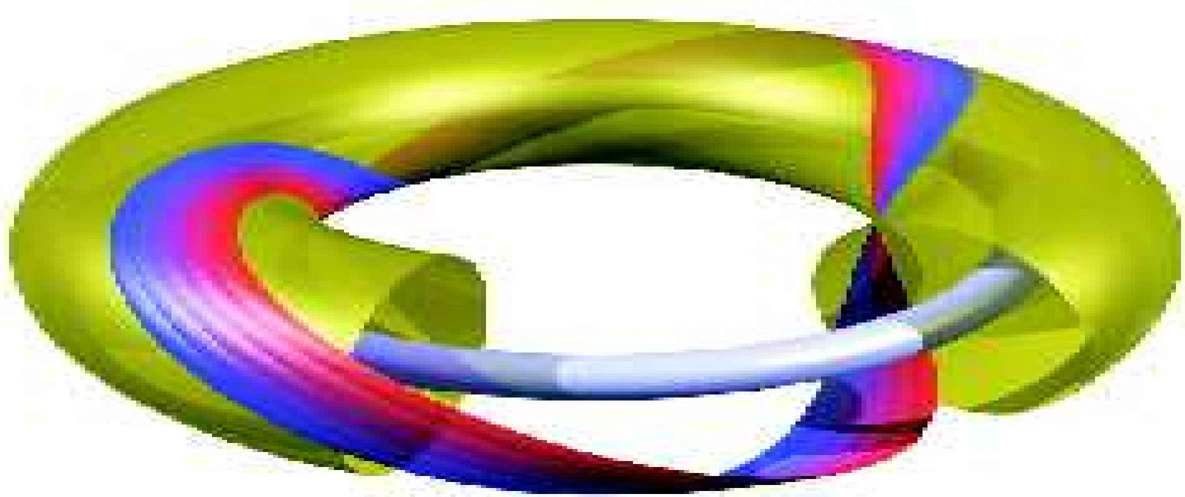
*FIAN, SC4, 22 May 2009*

# Goal

to construct classical solutions in gauge field theory – ideally in Standard Model – describing stationary flux loops



# Results

- Superconducting vortices in the Electroweak Theory

**M.S.V. Phys.Lett. B648, 249 (2007);**
**J.Garaud and M.S.V. Nucl.Phys. B799, 430 (2008);**
**hep-ph/0905.XXX**

- Making vortex loops in simpler models.

**E.Radu and M.S.V. Physics Reports, 468, 101-151 (2008)**
**Phys.Rev.DXX,xxx (2009);**
**in preparation**

# I. Superconducting electroweak vortices

generalizations of Z strings for non-zero currents

# Weinberg-Salam theory

$$\mathcal{L} = -\frac{1}{4g^2}\,\mathrm{W}^a_{\mu\nu}\mathrm{W}^{a\mu\nu} - \frac{1}{4g'^2}\,B_{\mu\nu}B^{\mu\nu} + (D_\mu\Phi)^\dagger D^\mu\Phi - \frac{\beta}{8}\left(\Phi^\dagger\Phi - 1\right)^2,$$

$$\mathrm{W}^a_{\mu\nu} = \partial_\mu \mathrm{W}^a_\nu - \partial_\nu \mathrm{W}^a_\mu + \epsilon_{abc}\mathrm{W}^b_\mu \mathrm{W}^c_\nu\,, \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,,$$

$$\Phi = \begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix}, \qquad D_\mu\Phi = \left(\partial_\mu - \frac{i}{2}\,B_\mu - \frac{i}{2}\,\tau^a \mathrm{W}^a_\mu\right)\Phi.$$

$$g = \cos\theta_W, \qquad g' = \sin\theta_W, \quad m_{\mathsf{z}} = 1/\sqrt{2},$$

$$m_{\mathsf{w}} = m_{\mathsf{z}}\cos\theta_W, \quad \beta = \left(\frac{m_{\mathsf{h}}}{m_{\mathsf{z}}}\right)^2 \qquad \boxed{1.5 \leq \beta \leq 3.5}$$

# Field equations

$$\partial_\mu B^{\mu\nu} = g'^2\,\Re(i\Phi^\dagger D^\nu\Phi)\,,$$

$$\partial_\mu \mathrm{W}_a^{\mu\nu} + \epsilon_{abc}\mathrm{W}_\sigma^b\mathrm{W}^{c\sigma\nu} = g^2\,\Re(i\Phi^\dagger\tau^a D^\nu\Phi)\,,$$

$$D_\mu D^\mu\Phi = \frac{\beta}{4}(\Phi^\dagger\Phi-1)\Phi.$$

$n^a = \Phi^\dagger\tau^a\Phi/(\Phi^\dagger\Phi) \Rightarrow$ electromagnetic, Z fields **/Nambu '77/**

$$F_{\mu\nu} = \frac{g}{g'}B_{\mu\nu} - \frac{g'}{g}n^a\mathrm{W}^a_{\mu\nu}\,, \qquad Z_{\mu\nu} = B_{\mu\nu} + n^a\mathrm{W}^a_{\mu\nu}\,,$$

$\Rightarrow$ electromagnetic current density

$$J_\mu = \partial^\nu F_{\nu\mu}$$

# Vortex symmetries

symmetry generators

$$K_{(t)} = \frac{\partial}{\partial t}, \qquad K_{(z)} = \frac{\partial}{\partial z}, \qquad K_{(\varphi)} = \frac{\partial}{\partial \varphi}$$

$\Rightarrow$ energy, momentum, angular momentum

$$\int T^0_\mu K^\mu_{(t)} d^2x, \qquad \int T^0_\mu K^\mu_{(z)} d^2x, \qquad \int T^0_\mu K^\mu_{(\varphi)} d^2x,$$

electric charge and current $(\alpha = 0, 3)$

$$\mathcal{I}^\alpha = \int J^\alpha d^2x$$

# Field ansatz

Symmetries commute $\Rightarrow \exists$ a gauge where the fields depend only on $\rho$. With $\sigma_\alpha = (\sigma_0, \sigma_3)$

$$\mathcal{W} = u(\rho)\,\sigma_\alpha dx^\alpha - v(\rho)\,d\varphi + \tau^1\,[u_1(\rho)\,\sigma_\alpha dx^\alpha - v_1(\rho)\,d\varphi]$$

$$+\,\tau^3\,[u_3(\rho)\,\sigma_\alpha dx^\alpha - v_3(\rho)\,d\varphi], \qquad \Phi = \begin{pmatrix} f_1(\rho) \\ f_2(\rho) \end{pmatrix}$$

- $\mathcal{W}_\rho = 0$ – gauge condition
- $\mathcal{W} = \mathcal{W}^*, \qquad \Phi = \Phi^*$
- Boosts along $z = x^3$ axis
- Residual global symmetry $(f_1 + if_2) \to e^{\frac{i}{2}\Gamma}(f_1 + if_2)$, $(u_1 + iu_3) \to e^{-i\Gamma}(u_1 + iu_3)$, $(W_1 + iW_3) \to e^{-i\Gamma}(W_1 + iW_3)$

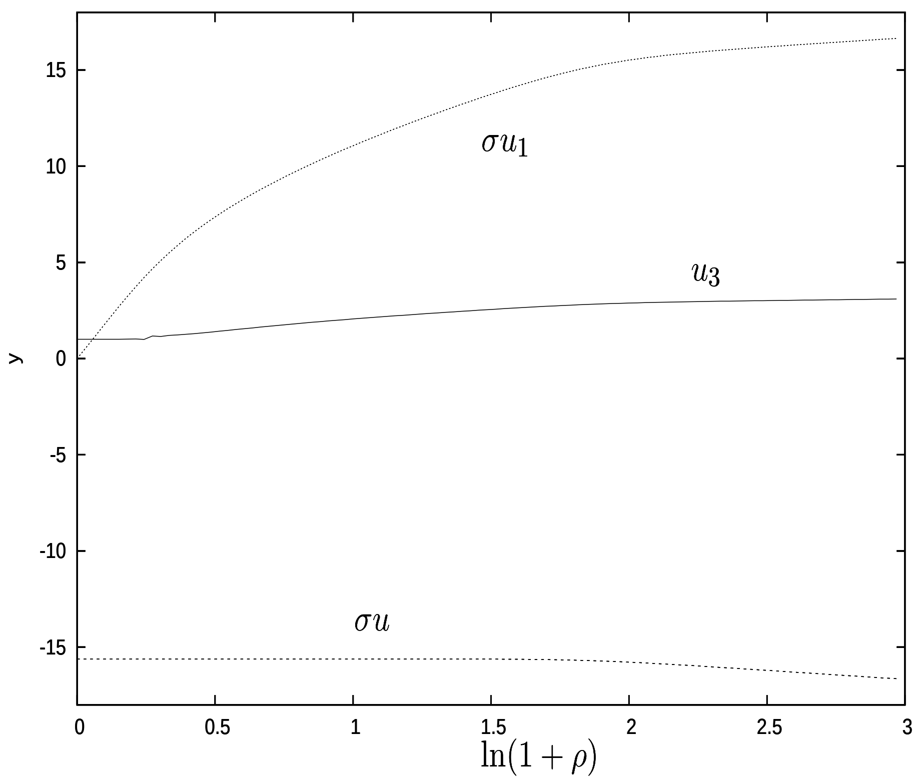
# Boundary conditions

- At the symmetry axis, $\rho = 0$, the fields are regular, energy density is finite.
- At infinity, $\rho \to \infty$, one has the Biot-Savart field of an infinitely long electric wire:

$$A_\mu = \frac{Q}{gg'}\,\sigma_\alpha dx^\alpha \ln\frac{\rho}{\rho_0} + C\,d\varphi$$

$$\Rightarrow Z_\mu = 0, \quad \mathbf{W}^\pm_\mu = 0, \quad \Phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

# Solutions



15
10
5
0
-5
-10
-15
y
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1
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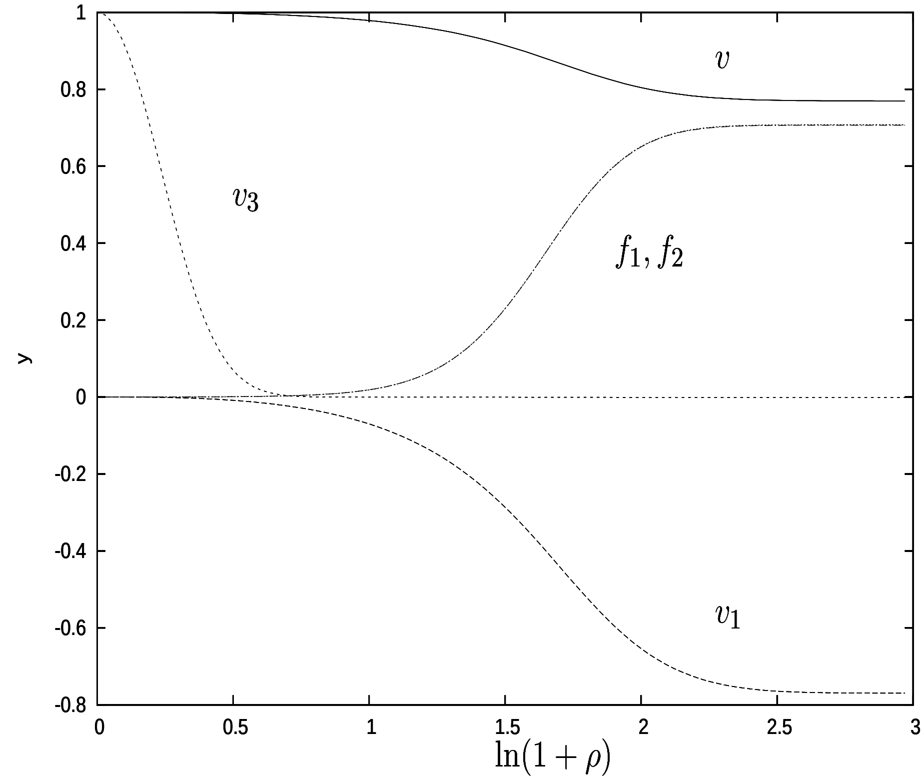
2
2.5
3
$\ln(1+\rho)$
$\sigma u_1$
$u_3$
$\sigma u$

1
0.8
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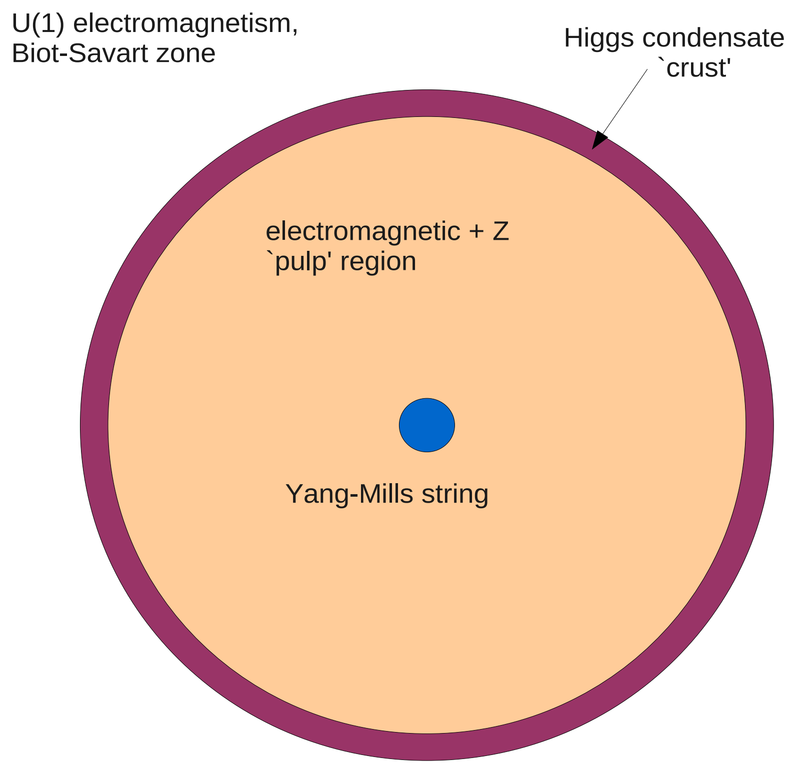
0.2
0
-0.2
-0.4
-0.6
-0.8
y
0
0.5
1
1.5
2
2.5
3
$\ln(1+\rho)$
$v$
$v_3$
$f_1, f_2$
$v_1$

# Superconducting electroweak vortices

are globally regular solutions, with a regular vortex core condaining massive W-condensate that creates a current. The current produces a Biot-Savart field outside the core.

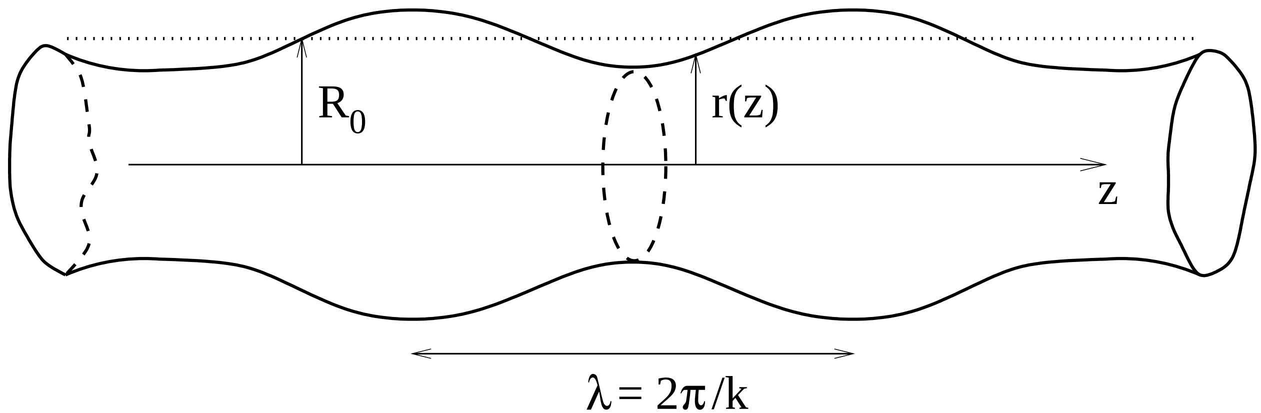
- Exist for any Higgs mass and for any $\sin^2\theta_{\rm W} \in [0,1]$
- When current tends to zero, they reduce to Z-strings.
- Their current can be arbitrarily large. For large currents they show the following structure.

# Vortex cross section



# Stability analysis for $\theta_w = \pi/2$

reveals only one negative mode, it is proportional to $\exp\{ikz\}$




$$\lambda > \lambda_{\mathrm{min}}(\mathcal{I})$$

$\Rightarrow$ it can be excluded by imposing periodic boundary conditions with period $L < \lambda_{\mathrm{min}}(\mathcal{I})$!

# Conclusion of the stability analysis

- Short vortex segments stable – no room to accommodate inhomogeneous unstable modes.
- Hydrodynamical analogy: Plateau-Rayleigh instability of a water jet, if the jet is long enough, ripples appear.
- Gravitational analogy: Gregory-Laflamme instability of a black string.

Perhaps small vortex loops are stable ?

# II. Explicit examples of ring solitons

**Eugen Radu and M.S.V.**

**Physics Reports, 468, 101-151 (2008)**

# Two types of ring solitons

## Knots

= twisted vortex loops stabilized by intrinsic deformations

## Vortons

= spinning vortex loops stabilized by the centrifugal force

# I. Faddeev-Skyrme model

O(3) sigma model with a Skyrme-type term, static energy

$$E[n^a] = \int \left( \frac{1}{4} (\partial_k n^a)^2 + \frac{1}{2} (F_{ik})^2 \right) d^3x$$

where $\sum_{a=1}^{3} n^a n^a = 1$ and $F_{ik} = \frac{1}{2} \epsilon_{abc} n^a \partial_i n^b \partial_k n^c$ .

Topological charge of Hopf, index of map $\mathbb{S}^3 \to \mathbb{S}^2$

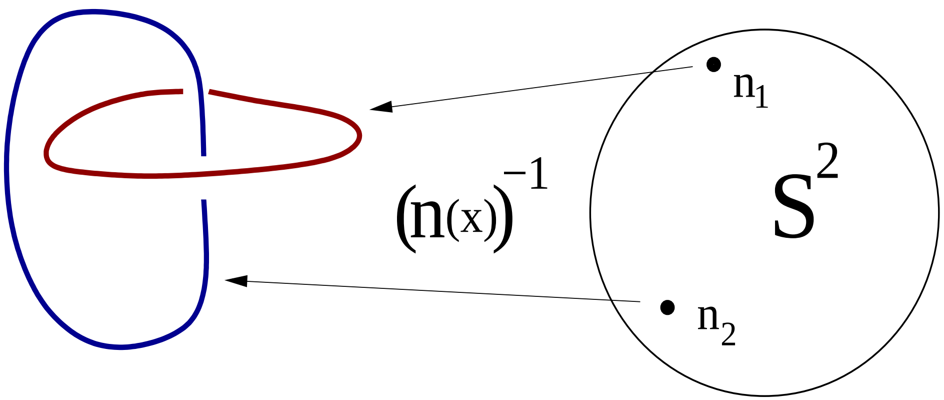
$$Q[n^a] = \frac{1}{16\pi^2} \int \epsilon_{ijk} A_i F_{jk} d^3x \in \mathbb{Z}$$

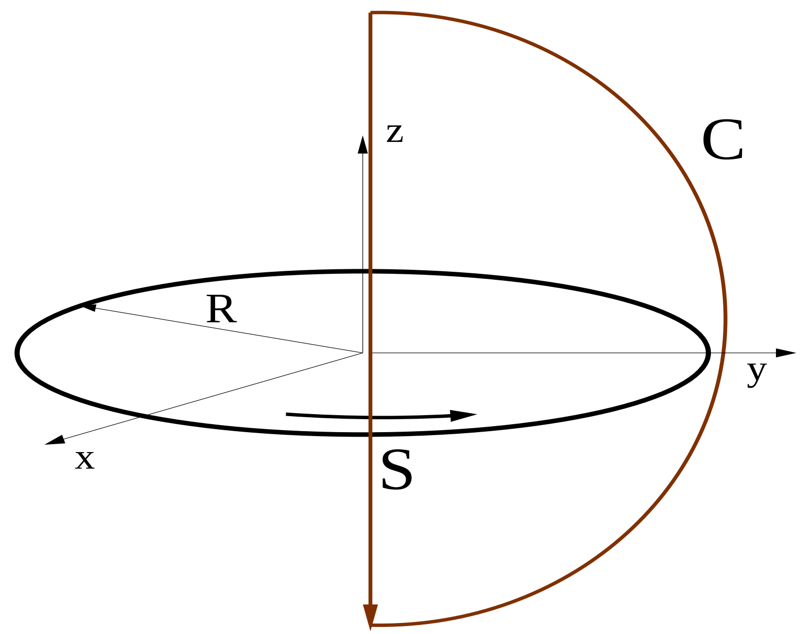
topological bound

$$E > c|Q|^{3/4}$$

**Vakulenko and Kapitansky '78**

# Hopf charge

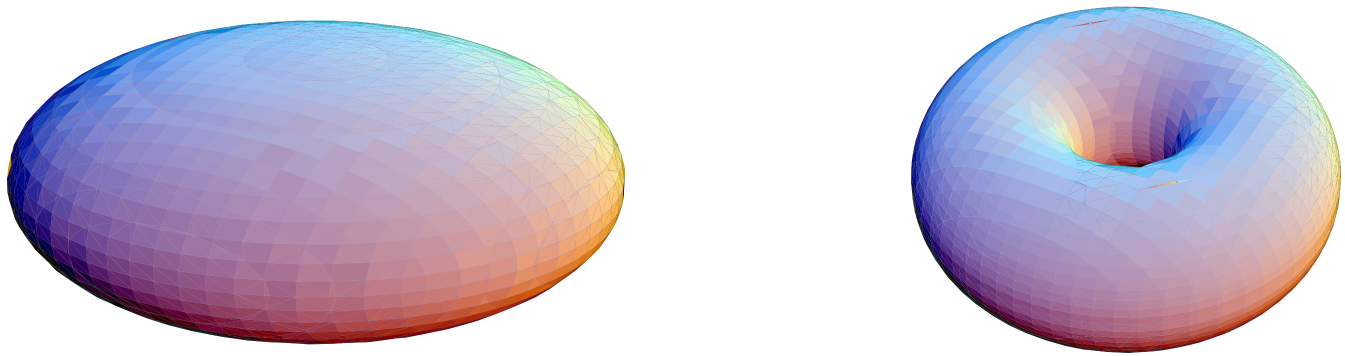



$$n^1 + in^2 = e^{i(m\varphi - n\psi)} \sin\Theta, \qquad n^3 = \cos\Theta$$

$$Q = mn$$

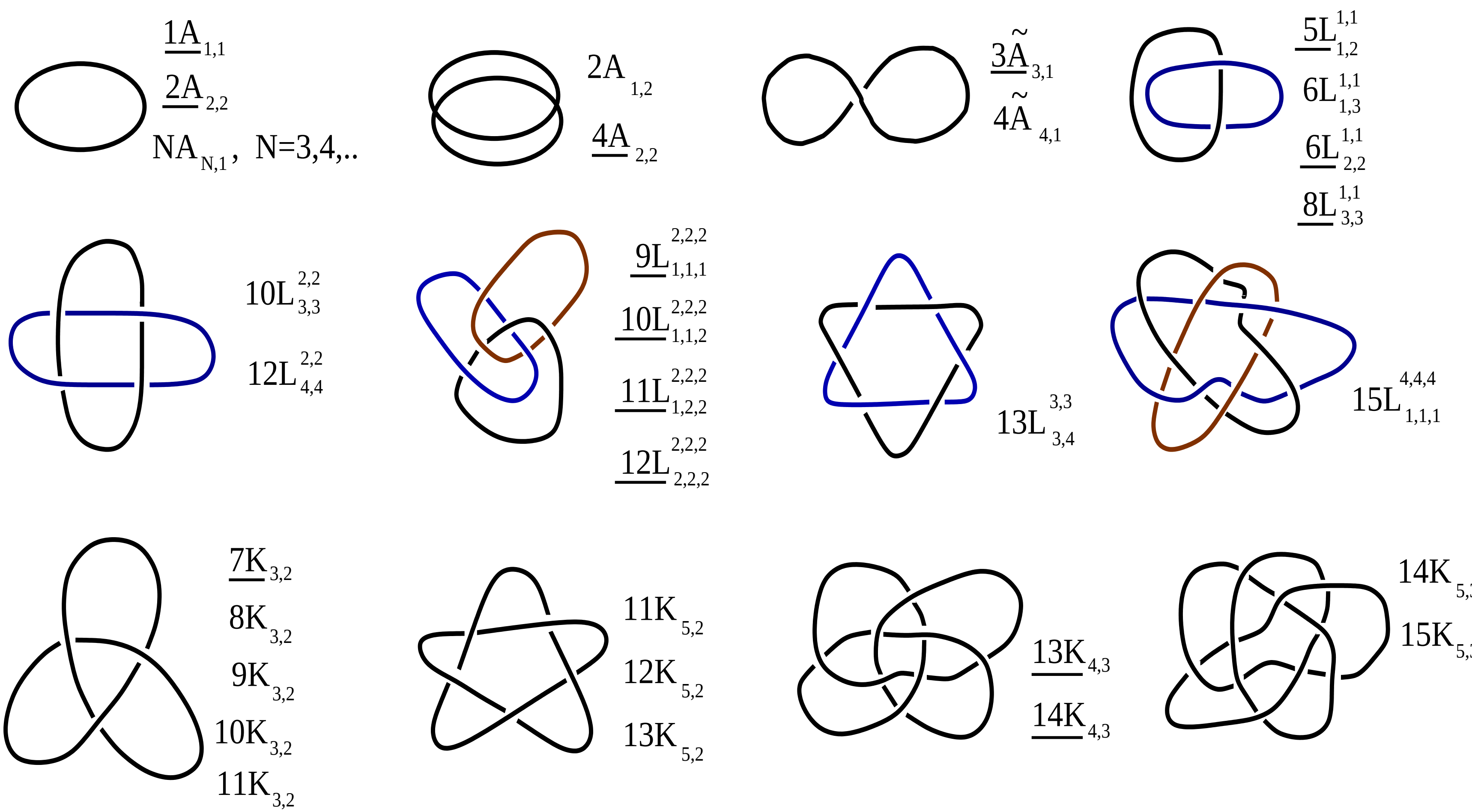
# Knots with $Q = 1, 2$



Gladikowski and Hellmund '97

Faddeev and Niemi '97

# Increasing the Hopf charge




/Battye and Sutcliffe/, /Hietarinta and Salo/, /R.Ward/

# II. Anomalous solitons

Abelian Higgs model, $\Phi \in \mathbb{C}^1$,

$$E[\Phi, A_k] = \int \left( |(\partial_k - iA_k)\Phi|^2 + \frac{1}{4}(F_{ik})^2 + \frac{\lambda}{4}(|\Phi|^2 - 1)^2 \right) d^3x \,,$$

Chern-Simons number

$$N_{\rm CS} = \frac{1}{4\pi^2} \int \epsilon_{ijk} \, A_i F_{jk} \, d^3x \,,$$

is fixed by fermion number **/Rubakov and Tavkhelidze '87/**.
Topological bound

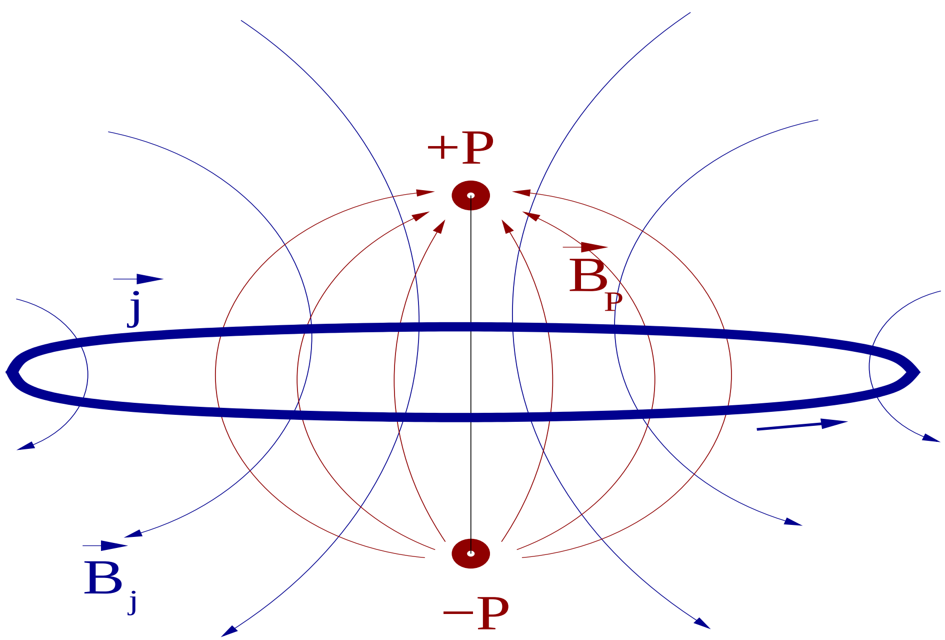
$$E[\Phi, A_k] \geq c|N_{\rm CS}|^{3/4}$$

**/Schmid and Shaposhnikov '07/**

# III. Magnetic monopole rings

$$L = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + D_\mu \Phi^a D^\mu \Phi^a - \frac{\lambda}{4} (\Phi^a \Phi^a - 1)^2$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon_{abc} A^b_\mu A^c_\nu, \quad D_\mu \Phi^a = \partial_\mu \Phi^a + \epsilon_{abc} A^b_\mu \Phi^c$$



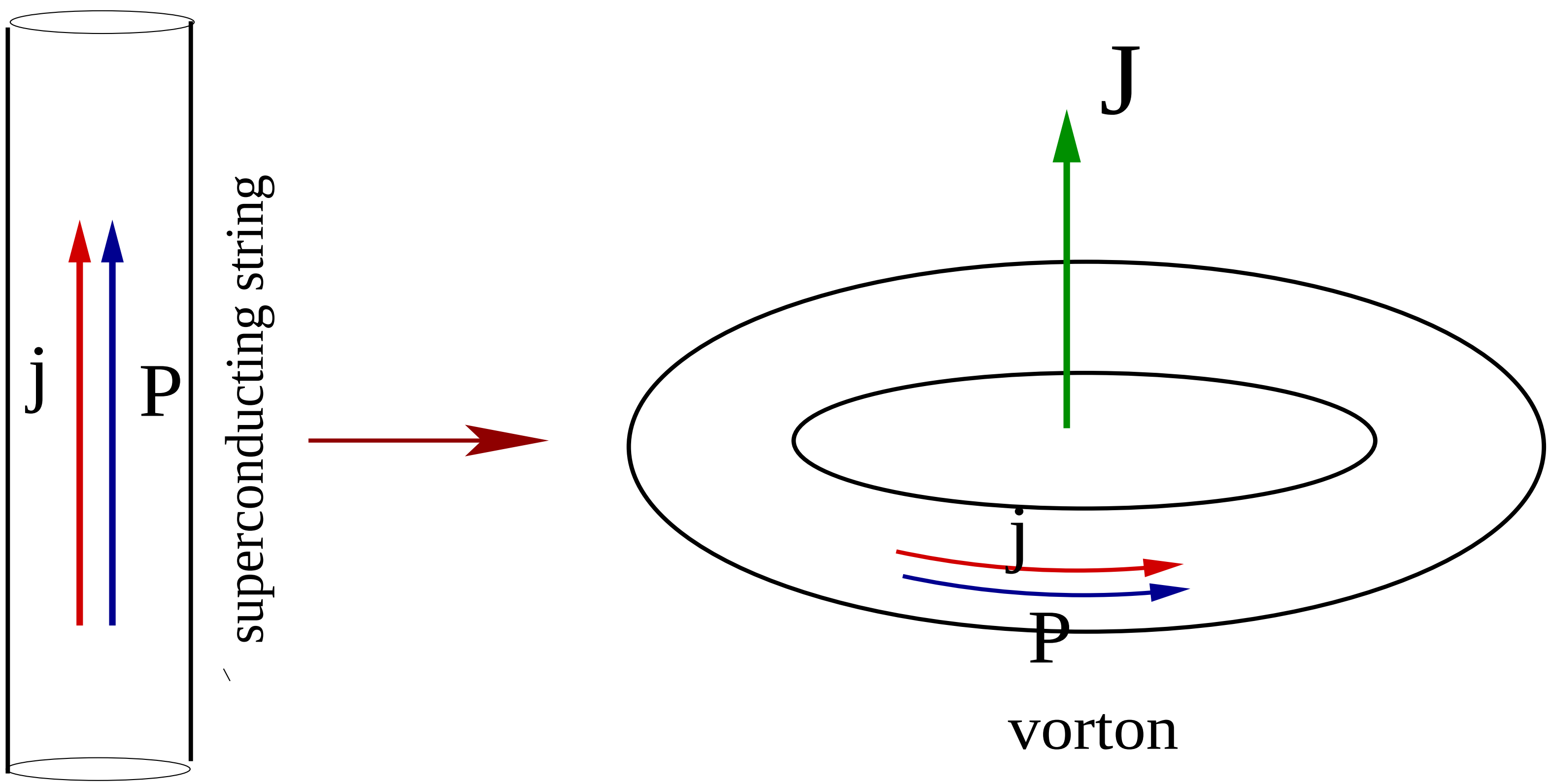

Kleihaus, Kunz, Shnir

# Spinning solitons

can one have such solutions ?

# Making a vorton

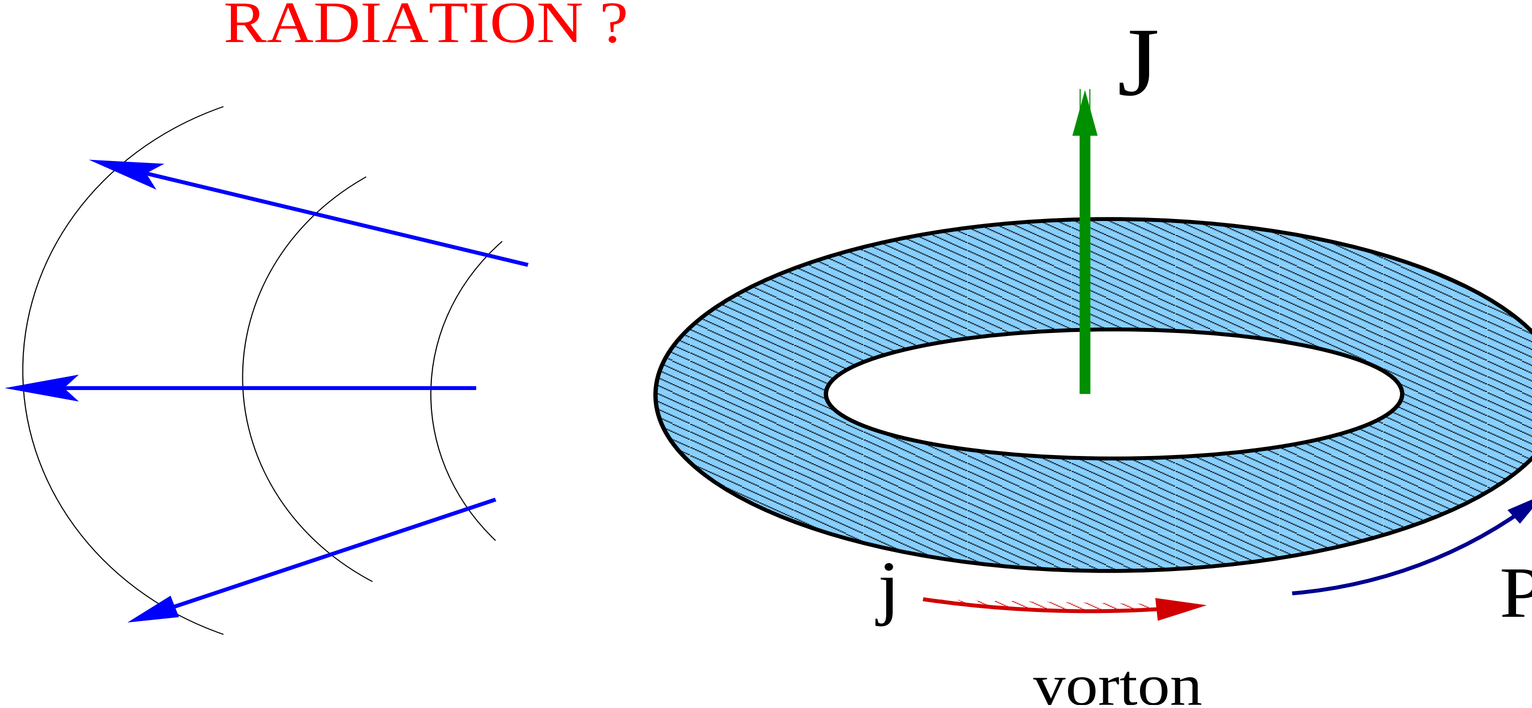
Effective macroscopic description:
superconducting vortex = elastic rope **/Davis, Shellard '88/**



# Problem

loop current = accelerated motion of charges $\Rightarrow$

RADIATION ?

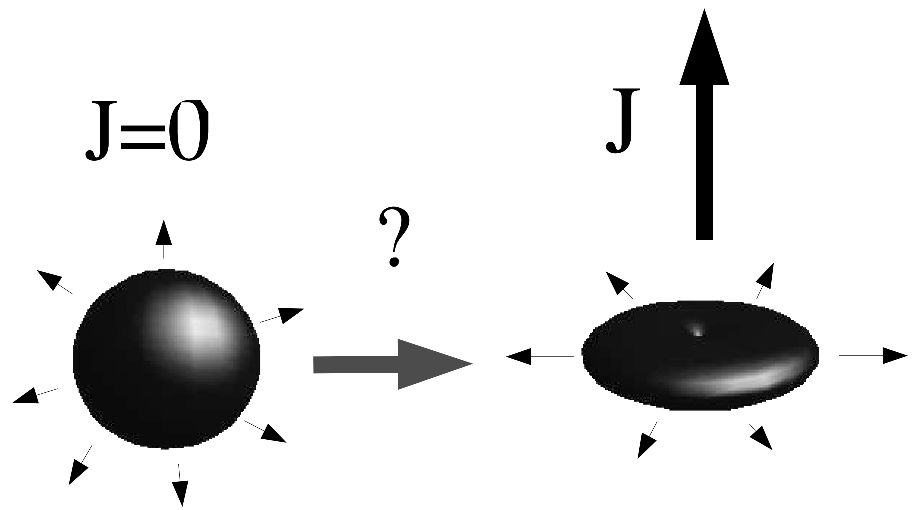



**Are they stationary ?**

# Another problem – a no-go result

$$J = \int T^0_\varphi d^3x = \oint \langle F^{0k} A_\varphi \rangle d_k S = 0$$

for all known SU(2) solitons: ’t Hooft-Polyakov monopoles, Julia-Zee dyons, sphalerons.




**Radu and Van der Bij ’01; M.S.V. and E.Woehnert ’03**

# Question

Are there stationary spinning field systems that do not radiate ?

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Are there stationary spinning field systems that do not radiate ?

## Answer

Yes, there are few examples in the literature

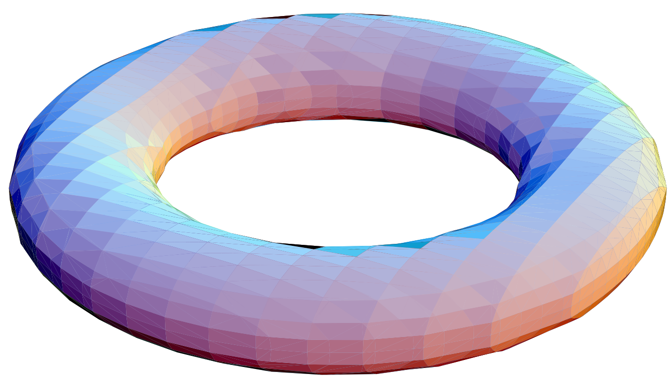
# IV. Spinning Q-balls

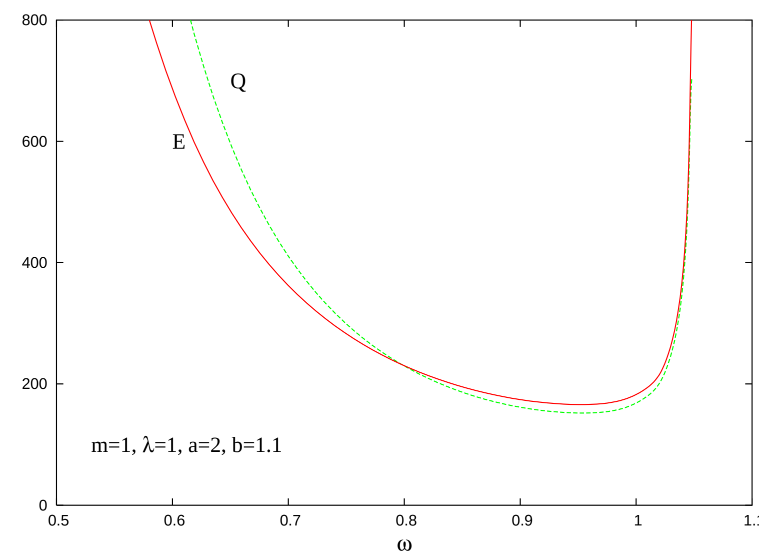
First known example of spinning solitons

$$L = \partial_\mu \phi^* \partial^\mu \phi - U(|\phi|), \qquad \phi = e^{i(\omega t + m\varphi)} f(r, \theta), \qquad J = \omega Q$$

**M.S.V. and E.Woehnert '02**

**Mihalache et al '02** – light bullets in non-linear optics






Twisted Q-balls **E.Radu and M.S.V. '08**

# V. Spinning Skyrmions

$$\mathcal{L} = \mathrm{tr}\left(\frac{1}{2}\,\partial_\mu U^\dagger \partial^\mu U + \frac{1}{8}\,[\partial_\mu U^\dagger, \partial_\nu U][\partial^\mu U^\dagger, \partial^\nu U]\right).$$

$$U = \left(\begin{array}{cc} \phi & i\sigma^* \\ i\sigma & \phi^* \end{array}\right),$$

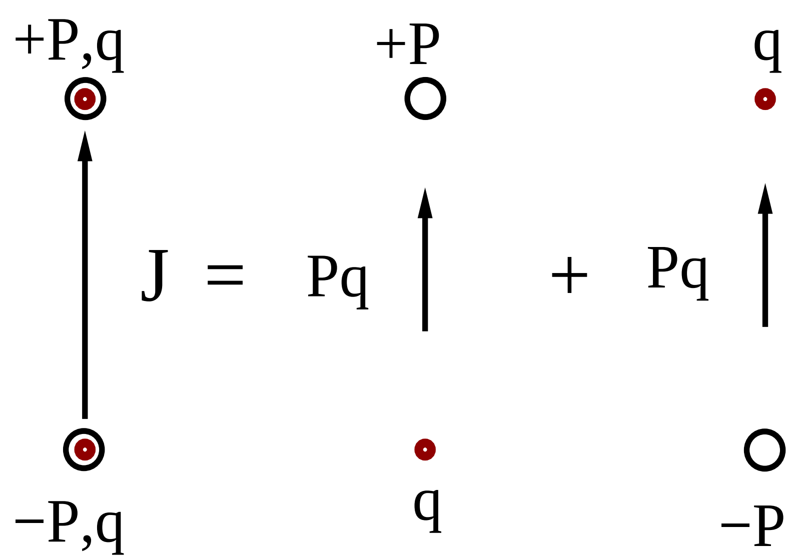
$$\phi = \cos\Theta(\rho,z)\,e^{i\psi(\rho,z)}, \qquad \sigma = \sin\Theta(\rho,z)\,e^{i(\omega t + m\varphi)}.$$

in $2+1$ **/Piette, Schroers and Zakrzewski '95/**
in $3+1$ **/Battye, Krusch and Sutcliffe '05/**

# VI. Rotating monopole-antimonopoles

$$L=-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+D_\mu\Phi^a D^\mu\Phi^a-\frac{\lambda}{4}(\Phi^a\Phi^a-1)^2$$

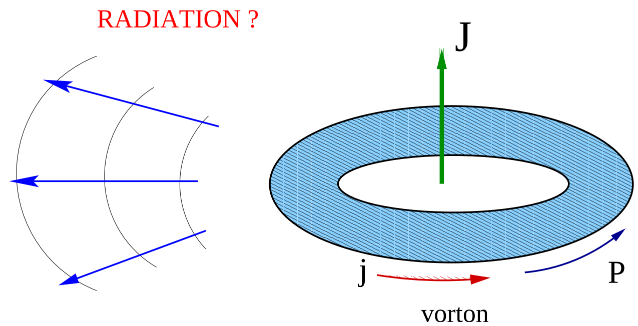
Taubes+electric charge= two dyons




Three dyons (not aligned) would give a stationary spinning system without axial symmetry

# Are vortons stationary ?

Dynamical simulations: **Lemperier and Shellard ’03**




Perhaps they radiate ???

# VII. First explicit vorton construction

**Eugen Radu and M.S.V. ’08**

# Vortons

Global limit of Witten's model

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \partial_\mu \sigma^* \partial^\mu \sigma - U$$

$$U = \frac{1}{4}\lambda_\phi(|\phi|^2 - \eta_\phi^2)^2 + \frac{1}{4}\lambda_\sigma |\sigma|^2(|\sigma|^2 - 2\eta_\sigma^2) + \gamma|\phi|^2|\sigma|^2,$$

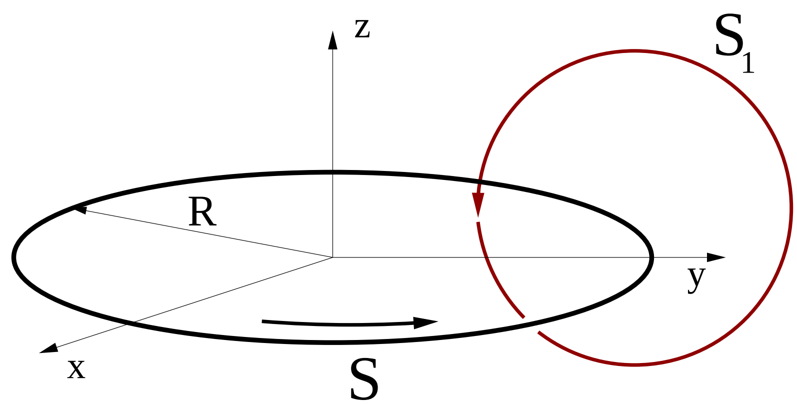
contains two Higgs with $M_\varphi = \sqrt{\lambda_\varphi}$ and $M_\sigma = \sqrt{\gamma - \frac{\lambda_\sigma}{2}\eta_\sigma^2}$ plus two Goldstones. Let

$$\phi = X(\rho, z) + iY(\rho, z) \equiv f(\rho, z)e^{i\psi(\rho,z)}, \qquad \sigma = Z(\rho, z)e^{i\omega t + im\varphi}$$

At large $r$

$$X = 1 + O(r^{-4}), \quad Y = O(r^{-2}), \quad Z \sim \exp(-\sqrt{M_\sigma^2 - \omega^2}\, r)$$

# Equations + boundary conditions




Phases of $\sigma$, $\phi$ increase by $2\pi m$ and $2\pi n$ along $S$ and $S_1$.

$$\Delta X = \left(\frac{\lambda_\phi}{2}(X^2+Y^2-1)+\gamma Z^2\right)X,$$

$$\Delta Y = \left(\frac{\lambda_\phi}{2}(X^2+Y^2-1)+\gamma Z^2\right)Y,$$

$$\Delta Z = \left(\frac{m^2}{r^2\sin^2\vartheta}-\omega^2+\frac{\lambda_\sigma}{2}(Z^2-\eta_\sigma^2)+\gamma(X^2+Y^2)\right)Z.$$

# Sigma model limit

$$\lambda_\sigma = \lambda_\phi = \beta, \quad \eta_\sigma = 1, \quad \gamma = \frac{1}{2}\beta + \gamma_0, \qquad \beta \to \infty \ ,$$

This enforces the constraint $X^2 + Y^2 + Z^2 = 1$ such that

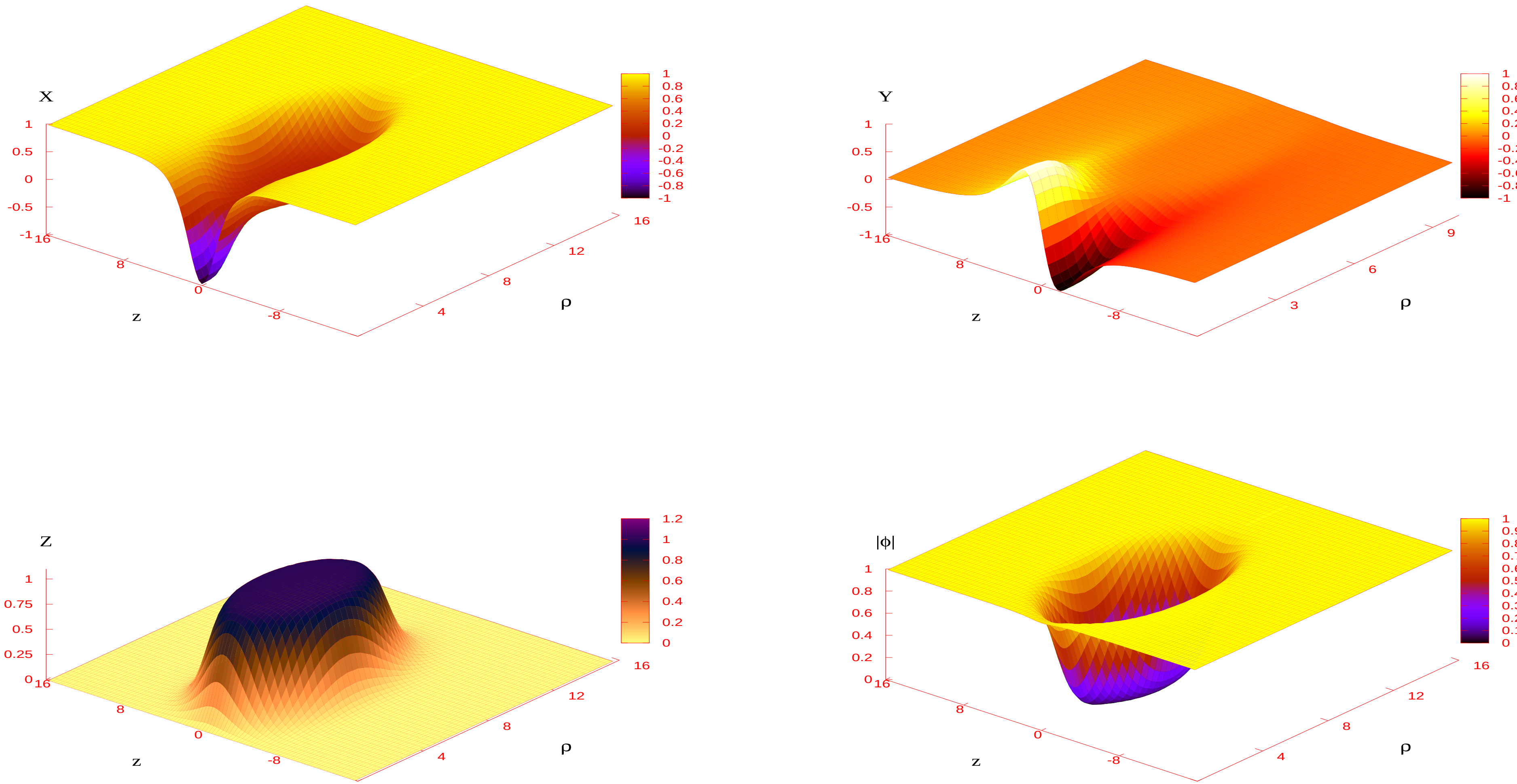
$$\Delta X = \left(Z^2 + \mu\right) X,$$

$$\Delta Y = \left(Z^2 + \mu\right) Y,$$

$$\Delta Z = \left(\frac{m^2}{r^2 \sin^2 \vartheta} - \omega^2 + X^2 + Y^2 + \mu\right) Z.$$

which are the Euler-Lagrang equations for the energy minimization problem studied by BCS.
Relaxing the sigma model condition gives the generic vortons.

# Solutions


X
1
0.5
0
-0.5
-1
16
8
0
-8
z
4
8
12
16
ρ
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Y
1
0.5
0
-0.5
-1
16
8
0
-8
z
3
6
9
ρ
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
Z
1
0.75
0.5
0.25
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16
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16
ρ
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0.8
0.7
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0.4
0.3
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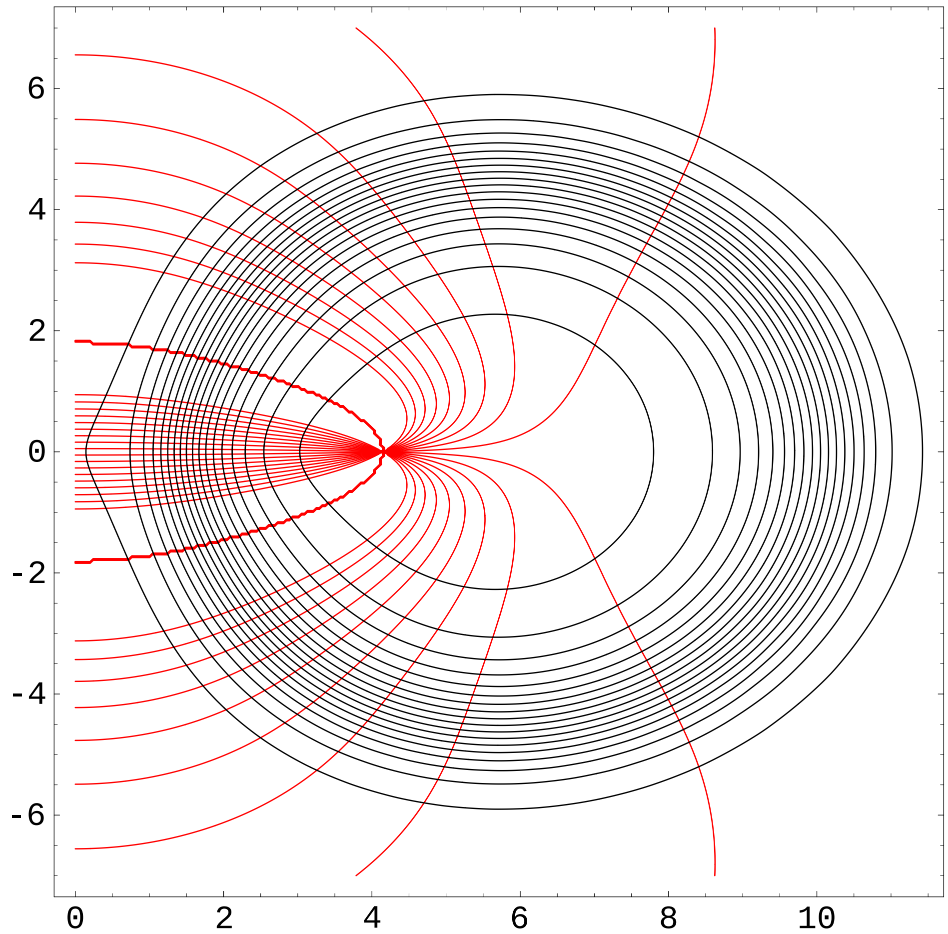
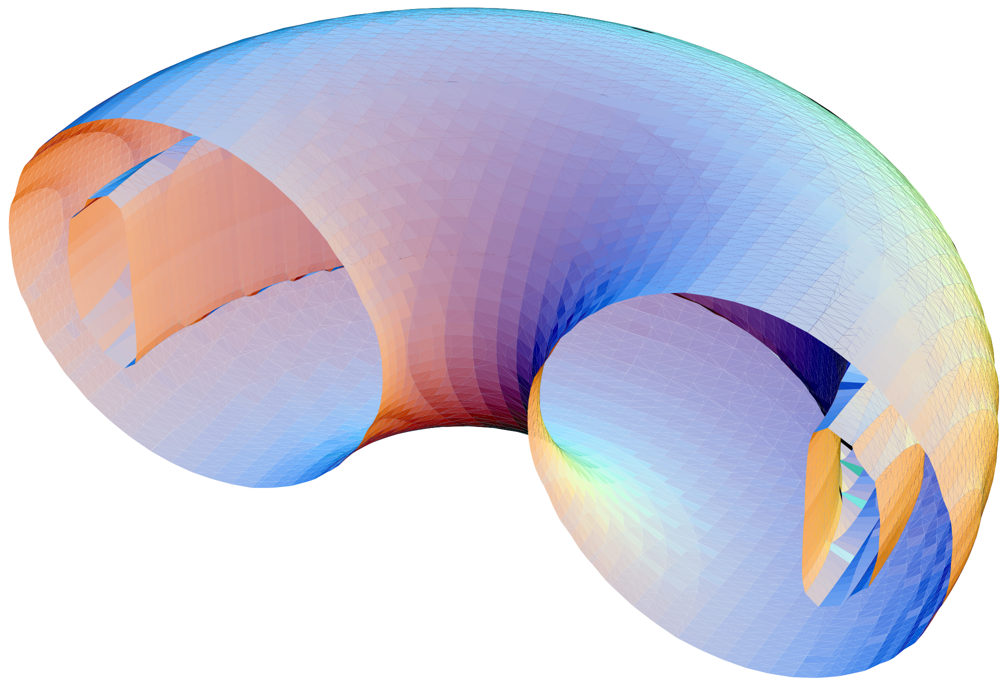
# Tube cross section



# Small $m$ – thick vortons



# VIII. Gauged vortons

$$\mathcal{L}_W = -\frac{1}{4}\sum_{a=1,2} F^{(a)}_{\mu\nu}F^{(a)\mu\nu} + D_\mu\phi^* D^\mu\phi + D_\mu\sigma^* D^\mu\sigma - U$$

with $F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu$ and

$$D_\mu\phi = (\partial_\mu - ig_1 A^{(1)}_\mu - ig_2 A^{(2)}_\mu)\phi$$

$$D_\mu\sigma = (\partial_\mu - ie_1 A^{(1)}_\mu - ie_2 A^{(2)}_\mu)\sigma$$

If $g_1 = g_2 = e_1 = e_2 = 0$: global vortons.
If $g_2 = e_1 = 0$: vortons in the full Witten model.
If $g_2 = e_2 = 0$: vortons in the Ginzburg-Landau model.

**E.Radu and M.S.V. '08**

# IX. Vortons in BE condensates

A vorton solution $\phi = \phi(\mathbf{x})$, $\sigma = \sigma(\mathbf{x})e^{i\omega t}$ solves the non-linear Schroedinger equation

$$i\frac{\partial \Psi_a}{\partial t} = \left(-\frac{1}{2}\,\Delta + \frac{1}{2}\sum_b \kappa_{ab}|\Psi_b|^2\right)\Psi_a$$

via

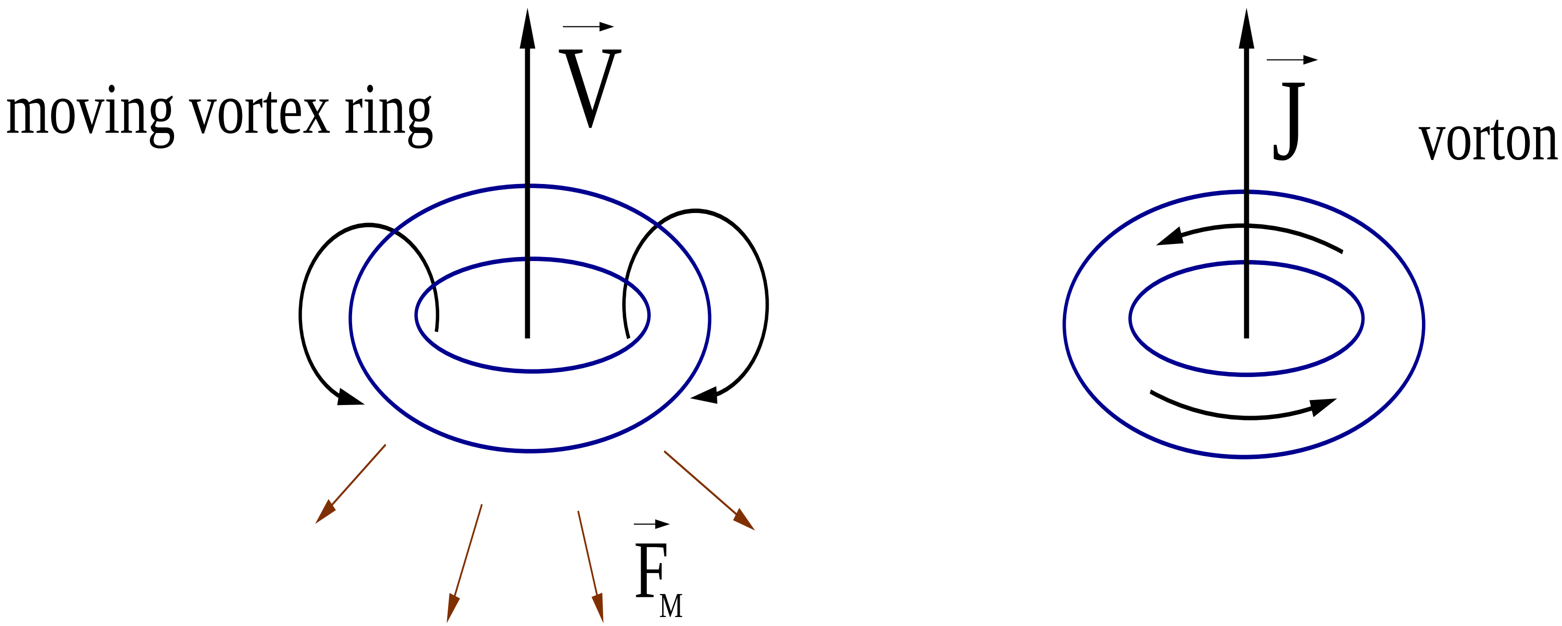
$$\Psi_1 = \; e^{-i\frac{\lambda_\phi}{4}\,t}\,\phi(\mathbf{x}), \qquad \Psi_2 = e^{-i\left(\frac{\lambda_\phi}{4}+\frac{\omega^2}{2}\right)t}\,\sigma(\mathbf{x})$$

if

$$\kappa_{11} = \frac{\lambda_\phi}{2}, \quad \kappa_{22} = \frac{\lambda_\sigma}{2}, \quad \kappa_{12} = \kappa_{21} = \gamma$$

Solutions of NSE can be obtained via the energy minimization BCS ’02

# Moving vortex rings




Smoke rings, hydrodynamical rings /Kelvin, Helmholtz/

# X. Rings in condenced media

In BEC /Jones and Roberts '74, Berloff/

$$2i\frac{\partial\Psi}{\partial t} = \left(-\Delta + \kappa(|\Psi|^2 - 1)\right)\Psi\,.$$

$$\Psi = \Psi(\rho, z - Vt),$$

In ferromagnetics /Cooper '99, Sutcliffe '06/

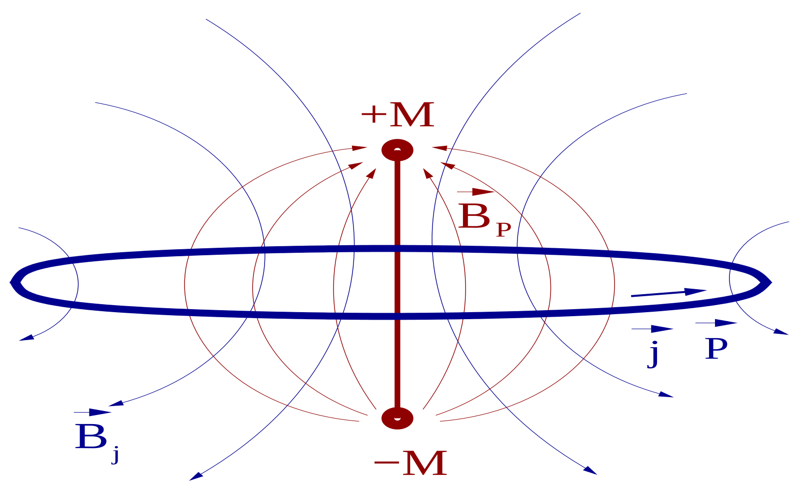
$$\frac{\partial n^a}{\partial t} = \epsilon_{abc} n^b \Delta n^c.$$

$$n^1 + in^2 = \sin\Theta(\rho, z - Vt)\exp\{i[m\varphi + \omega t - \psi(\rho, z - Vt)]\},$$

$$n^3 = \cos\Theta(\rho, z - Vt),$$

# XI. Spinning sphalerons

Stationary generalizations of the static electroweak sphaleron solution. Exist for any $\theta_{\mathrm{W}}$, if $\theta_{\mathrm{W}} \neq 0$ then $J \sim Q$. For large $J$ show the Regge behavior, $J \sim E^2$, contain a string loop




Their action decreases with energy.

**Eugen Radu and M.S.V. Phys.Rev.D (March 2009)**

# Summary

- New solutions describing superconducting vortices in Weinberg-Salam theory are constructed.
- Their stability is analized – short vortex pieces are perturbatively stable, so that that short vortex loops could perhaps also be stable.
- Solutions describing stationary vortex loops are constructed within the U(1)$\times$U(1) global and local field theory models.
- Other new stationary spinning solitons obtained: $Q$-balls and electroweak sphalerons.
- A review of known stationary ring solitons is given.