Inside the BTZ black hole

M. O. Katanaev
Steklov Mathematical Institute, Moscow

de Berredo-Peixoto, Katanaev Phys. Rev. D75(2007)024004

$\mathbb{M}, \quad x^\mu, \ \mu = 0,1,2$ - local coordinates on space-time $\mathbb{M}$

$g_{\mu\nu}(x)$ - Lorentzian signature metric $\operatorname{sign} g_{\mu\nu} = (+--)$

Einstein's equations: $\boxed{R_{\mu\nu} = \Lambda g_{\mu\nu}}$

$\Lambda = -\frac{2}{l^2}$ - negative cosmological constant

$R_{\mu\nu\rho\sigma} = -\varepsilon_{\mu\nu\lambda}\varepsilon_{\rho\sigma\varsigma}R^{\lambda\varsigma}$ - curvature tensor

$\varepsilon_{\mu\nu\lambda}$ - totally antisymmetric third rank tensor

Smooth solutions:

$$\mathbb{M} = \frac{\mathbb{U}}{\Gamma}$$

$$\mathbb{U} = \begin{cases} dS, & \Lambda > 0, \quad \text{- de Sitter space-time} \\ \mathbb{R}^{1,2}, & \Lambda = 0, \quad \text{- Minkowskian space-time} \\ AdS, & \Lambda < 0. \quad \text{- Anti de Sitter space-time} \end{cases}$$

The BTZ solution

Banados, Teitelboim, Zanelli Phys.Rev.Lett. 69 (1992)1849

$$ds^2 = \left(-M + \frac{J^2}{4r^2} + \frac{r^2}{l^2} \right) dt^2 - \frac{dr^2}{-M + \dfrac{J^2}{4r^2} + \dfrac{r^2}{l^2}} - r^2 \left(d\varphi - \frac{J}{2r^2} dt \right)^2$$

$M > 0, \quad J$ - two integration constants (mass and angular momentum)

$t \in (-\infty, \infty), \quad r \in (0, \infty), \quad \varphi \in (0, 2\pi)$ - cylindrical coordinates

$K_1 = \partial_{t,} \quad K_2 = \partial_\varphi$ - two commuting Killing vector fields

$$-M + \frac{J^2}{4r^2} + \frac{r^2}{l^2} = 0 \quad \Rightarrow \quad r_\pm^2 = \frac{Ml^2}{2} \left(1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}} \right) \qquad |J| < Ml$$

$r_+, \quad r_-$ - outer and inner horizons, $r_3 = Ml^2$

	t	r	φ
$r_3 < r < \infty :$	$+$	$-$	$-$
$r_+ < r < r_3 :$	$-$	$-$	$-$
$r_- < r < r_+ :$	$-$	$+$	$-$
$0 < r < r_- :$	$-$	$-$	$-$

$\mathbb{M}$ - is not a manifold, because
points at $r\_$ do not have neiborhoods

$r = 0$ - is regular

The interior region $\Lambda = 0$

The limit: $l \to \infty, \quad r_3 \to \infty, \quad r_+ \to \infty$

$$\boxed{ds^2 = -\alpha^2 dt^2 - \frac{dr^2}{-\alpha^2 + \dfrac{c^2}{r^2}} - r^2 d\varphi^2 + 2cd\varphi dt}$$

$$\alpha^2 = M, \quad c = \frac{J}{2}$$

$$t = t' + \frac{c}{\alpha^2}\varphi$$ - coordinate transformation

$$ds^2 = -\alpha^2 dt'^2 - \frac{dr^2}{-\alpha^2 + \dfrac{c^2}{r^2}} - \left(r^2 - \frac{c^2}{\alpha^2}\right) d\varphi^2$$ - diagonal form

Global solution = maximally extended along geodesics

The space-time: $\mathbb{M} = \mathbb{R} \times \mathbb{U}$ $\qquad t' \in \mathbb{R}, \quad (r, \varphi) \in \mathbb{U}$

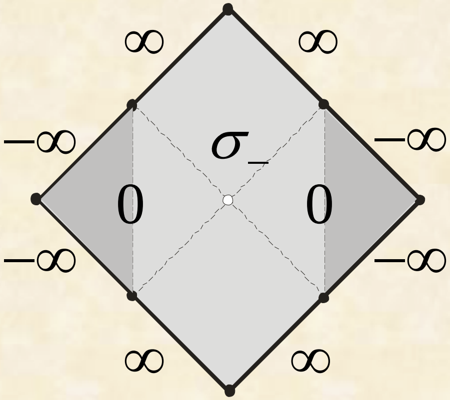
Coordinate transformations

$$r = \sqrt{2\alpha\sigma}, \quad \sigma > 0, \quad \alpha > 0 \qquad\qquad r = \sqrt{2\alpha\sigma}, \quad \sigma < 0, \quad \alpha < 0$$

$$dl^2 = -\frac{\alpha^2}{c^2 - 2\alpha^3\sigma} d\sigma^2 + \frac{c^2 - 2\alpha^3\sigma}{\alpha^2} d\varphi^2$$ - the induced metric on $\mathbb{U}$

Consider the case $\boxed{\varphi \in (-\infty, \infty)}$

$$\sigma_- \leftrightarrow r_-, \quad \sigma_- = \frac{c^2}{2\alpha^3}$$

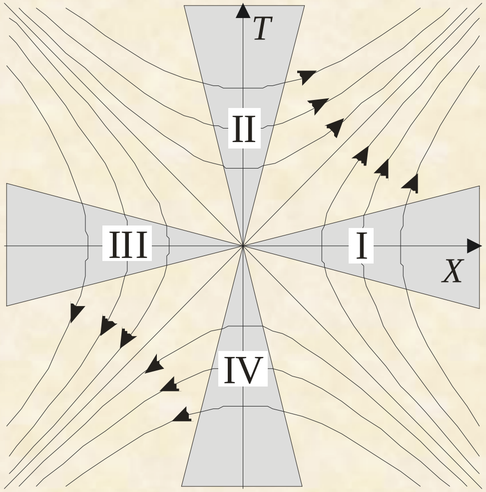


- the Carter-Penrose diagram for the Minkowskian space time $\mathbb{R}^{1,1}$

The space-time: $\mathbb{M} = \mathbb{R} \times \mathbb{R}^{1,1} = \mathbb{R}^{1,2}$

Four copies of the BTZ solution $\underbrace{\text{cover}}_{\text{for } \Lambda=0}$ the whole Minkowskian space-time $\mathbb{R}^{1,2}$

Transformation to Cartezian coordinates



Minkowskian space-time $\mathbb{R}^{1,1}$

$$dl^2 = dT^2 - dX^2 = dR^2 - R^2 d\Phi^2$$

Transformation to polar coordinates:

$$\text{I:}\quad T = R\sinh\Phi, \quad X = R\cosh\Phi,$$

$$\text{II:}\quad T = R\cosh\Phi, \quad X = R\sinh\Phi,$$

$$\text{III:}\quad T = -R\sinh\Phi, \quad X = -R\cosh\Phi,$$

$$\text{IV:}\quad T = -R\cosh\Phi, \quad X = -R\sinh\Phi,$$

Transformation of coordinates:

$$\text{II, IV:}\qquad R = \frac{\sqrt{2\alpha^3\sigma - c^2}}{\alpha^2}, \quad \sigma > \frac{c^2}{2\alpha^3}$$

$$\Phi = \alpha\varphi$$

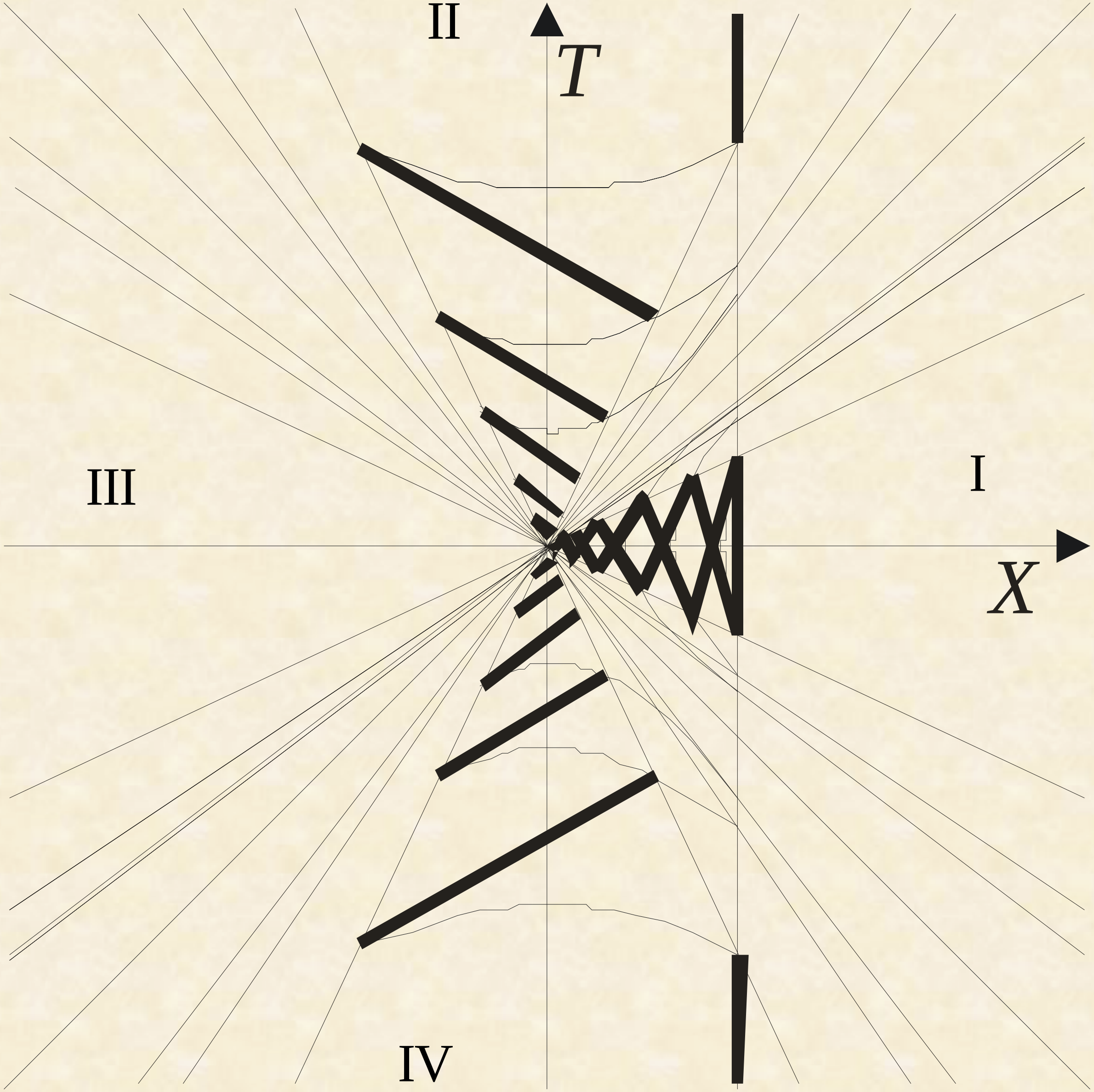
Periodicity of the angle:

$$\boxed{\varphi \sim \varphi + 2\pi \quad \Leftrightarrow \quad \Phi \sim \Phi + 2\pi\alpha}$$

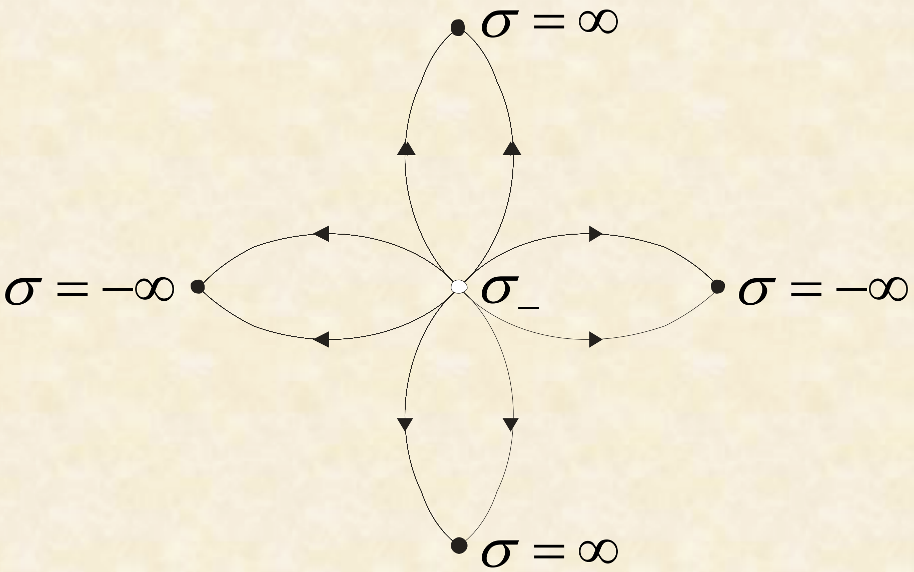
Transformation group (the isometry) $\mathbb{G}: \quad \Phi \to \Phi + 2\pi\alpha$

The interior region
of the BTZ black hole $\mathbb{M} = \dfrac{\mathbb{R}^{1,1}}{\mathbb{G}}$ is not a manifold

Geodesics



The Carter-Penrose diagram



The space-time $\mathbb{M}=\dfrac{\mathbb{R}^{1,1}}{\mathbb{G}}$

$$\mathbb{G}:\quad \Phi\to\Phi+2\pi\alpha$$

Euclidean version

The limit: $l \to \infty, \quad r_+ \to \infty$

For the interior region

$$t \to iz, \quad \varphi \to i\varphi$$

$$ds^2 = \alpha^2 dz^2 + \frac{dr^2}{\alpha^2 - \frac{c^2}{r^2}} + r^2 d\varphi^2 - 2c\, d\varphi\, dz$$

$$r_- < r < \infty, \quad \operatorname{sign} g_{\mu\nu} = (+++)$$

$$0 < r < r_-, \quad \operatorname{sign} g_{\mu\nu} = (+--)$$

where $r_- = \dfrac{c}{\alpha}$

The outer region $r_- < r < \infty$

The Euclidean space $\mathbb{R}^3: \; ds^2 = dX^2 + dY^2 + dZ^2 = dR^2 + R^2 d\Phi^2 + dZ^2$

The coordinate transformation

$$R = \frac{r}{\alpha}\sqrt{1-\frac{r_-^2}{r^2}}, \qquad r_- < r < \infty,$$

$$\Phi = \alpha\varphi, \qquad 0 < \Phi < 2\pi\alpha,$$

$$Z = \alpha z - r_-\varphi, \qquad -\infty < z < \infty.$$

$$\boxed{ds^2 = \alpha^2 dz^2 + \frac{dr^2}{\alpha^2 - \frac{c^2}{r^2}} + r^2 d\varphi^2 - 2c\, d\varphi\, dz}$$

The deficit angle $2\pi\theta$ of a conical singularity in the R, Φ plane

$$\theta = \alpha - 1$$

The outer region $r_- < r < \infty$

The Euclidean space $\mathbb{R}^3: \; ds^2 = dX^2 + dY^2 + dZ^2 = dR^2 + R^2 d\Phi^2 + dZ^2$

The coordinate transformation
$$R = \frac{f}{\alpha}, \qquad 0 < f < \infty,$$
$$\Phi = \alpha\psi, \qquad 0 < \Phi < 2\pi\alpha,$$
$$Z = \varsigma - c\psi, \qquad -\infty < \varsigma < \infty.$$

$f = f(\rho)$ - the radial coordinate

Coordinates f, ψ, ς cover the whole Euclidean space $\mathbb{R}^3$ and nothing else

$$\boxed{ds^2 = \frac{df^2}{\alpha^2} + (f^2 + c^2)d\psi^2 + d\varsigma^2 - 2c\,d\varsigma\,d\psi}$$

For $c = 0$ we have a conical singularity in each section $\varsigma = \text{const}$

The deficit angle $2\pi\theta, \qquad \theta = \alpha - 1$

The interior region $0 < r < r_-$ covers the Minkowskian space-time $\mathbb{R}^{1,2}$

Solid state physics interpretation

Geometric theory of defects: Katanaev, Volovich. Ann.Phys. 216(1992)1; ibid.271(1999)203,
Katanaev. Theor.Math.Phys. 135(2003)733; ibid.138(2004)163,
Katanaev. Phys.Usp.48(2005)675

Elastic deformations = diffeomorphisms of the Euclidean space $\mathbb{R}^3$
Dislocations = nontrivial torsion = surface density of Burgers vector
Disclinations = nontrivial curvature = surface density of Frank vector

Basic variables: triad field $e_\mu{}^i$ and SO(3)-connection $\omega_\mu{}^{ij}$

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_{\nu j} - (\mu \leftrightarrow \nu)$$ - torsion

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^j - (\mu \leftrightarrow \nu)$$ - curvature

The action:

$$\boxed{L = \kappa \tilde{R} - \gamma R_{[ij]} R^{[ij]}}$$

$\tilde{R}(e)$ - the Hilbert-Einstein action

$R_{[ij]}(e,\omega)$ - antisymmetric part of Ricci tensor

Absence of disclinations: $R_{\mu\nu\rho}{}^\sigma = 0 \ \rightarrow$ SO(3) -connection is a pure gauge

The geometry is given by the triad field $e_\mu{}^i$

The metric $g_{\mu\nu} = e_\mu{}^i e_\nu{}^j \delta_{ij}, \quad \delta_{ij} = \mathrm{diag}(+++)$ satisfies Einstein's equations

$$\boxed{R_{\mu\nu} = T_{\mu\nu}}$$

The elastic gauge

Flat Euclidean metric in $\mathbb{R}^3:\quad ds^2=\hat{g}_{\mu\nu}dx^\mu dx^\nu=d\rho^2+\rho^2 d\psi^2+d\varsigma^2$

$$\boxed{(1-2\sigma)\hat{g}^{\mu\nu}\hat{\nabla}_\mu e_{\nu i}+\sigma\hat{e}^\mu{}_i\hat{\nabla}_\mu e^T=0}$$

- the elastic gauge

$\sigma=\mathrm{const}$ - the Poisson ratio $\qquad -1\le\sigma\le\dfrac{1}{2}$

$$e^T=\hat{e}^\mu{}_i e_\mu{}^i$$

← In the absence of defects and for small deformations

$$e_{\mu i}=\delta_{\mu i}-\frac{1}{2}(\partial_\mu u_i+\partial_i u_\mu)$$

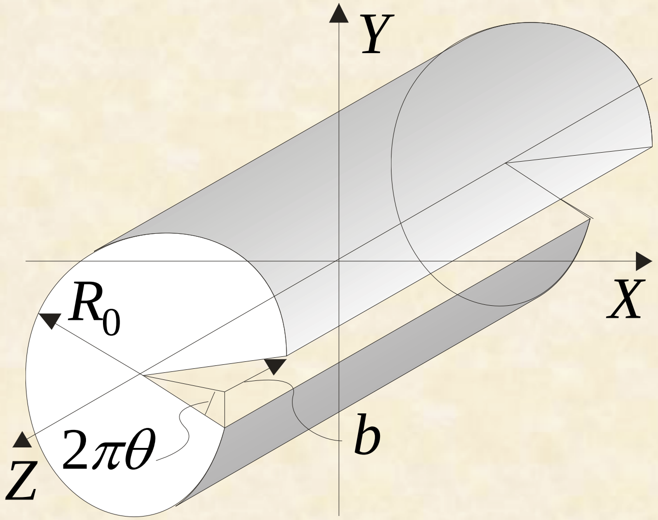
u^i - displacement vector field

$\dfrac{1}{2}(\partial_\mu u_i+\partial_i u_\mu)$ - the strain (deformation) tensor

$$\boxed{(1-2\sigma)\Delta u_i+\partial_i\partial_j u^j=0}$$

- elasticity theory equations

Euclidean BTZ solution



- combined wedge and screw dislocations

$$ds^2 = \frac{df^2}{\alpha^2} + (f^2 + c^2)d\psi^2 + d\varsigma^2 - 2c\,d\varsigma\,d\psi$$

- the Euclidean BTZ solution in the outer region

$$0 < f < \infty, \;\; 0 < \psi < 2\pi, \;\; -\infty < \varsigma < \infty$$

The elastic gauge: $f \to \rho$

Comparison with elasticity theory

$$ds^2 = \left(\frac{\rho}{R_0}\right)^{2(\gamma-1)} d\rho^2 + \left(\frac{\alpha^2}{\gamma^2}\left(\frac{\rho}{R_0}\right)^{2(\gamma-1)} \rho^2 + c^2\right) d\psi^2 + d\varsigma^2 - 2c\,d\varsigma\,d\psi$$

$$\gamma = -\theta B + \sqrt{\theta^2 B^2 + 1 + \theta}, \qquad B = \frac{\sigma}{2(1-\sigma)}$$

for $\theta \ll 1$

$$ds^2_{(\text{elastic})} = \left(1 + \theta\frac{1-2\sigma}{1-\sigma}\ln\frac{\rho}{R_0}\right) d\rho^2 +$$

$$+\left(\rho^2\left(1 + \theta\frac{1-2\sigma}{1-\sigma}\ln\frac{\rho}{R_0} + \theta\frac{1}{1-\sigma}\right) + c^2\right) d\psi^2 + d\varsigma^2 - 2c\,d\varsigma\,d\psi$$

Elasticity theory - small relative deformations: $\partial_\mu u^i \ll 1: \quad \theta \ll 1, \ \frac{b}{R_0} \ll 1, \ \rho \sim R_0$

Geometric theory of defects:
the metric is simpler, valid everywhere and for all $\theta, \ b$

Conclusion

1) For BTZ solution we have singularity in the manifold itself at r_-

2) The geodesics can be continued through r_-

3) In the Euclidean version of the BTZ solution, the space-time breaks into disconnected manifolds along r_-

4) The point $r = 0$ is regular.

5) For zero cosmological constant the Euclidean BTZ solution has straightforward interpretation in solid state physics describing combined wedge and screw dislocation.

6) The elasticity result reproduces only the linear approximation to the exact solution of Einstein equations.

7) The result (metric) can be measured experimentally.