A Fibre Approach to Harmonic Analysis Of Higher-Spin Field Equations

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(C. I., Per Sundell, JHEP, 0810:022, 2008)

Moscow, Fourth Sakharov Conference, May 18 2009

Why Higher Spins?

1. Crucial problem in Field Theory

2. Key role in String Theory
   - Strings beyond low-energy SUGRA
   - HSGT as symmetric phase of String Theory?

3. Positive results from AdS/CFT

Summary

- Field Theory: Unfolded formulation

- Group Theory: (U)IRs of $\mathfrak{so}(D-1,2)$

- Link: 1) Lorentz-covariant ↔ Compact slicings
  2) Operator ↔ state correspondence

  Harmonic analysis in fibre due to unfolding's Dynamics/Fibre "duality"

- Conclusions & Outlook

$\boxed{\text{Focus on AdS bosonic model}}$

Unfolded Formulation

(Vasiliev, '89)
(Chevalley-Eilenberg, Sullivan ,D'Auria-Fré…)

- Unfolding = formulating dynamics via consistent ($d^2=0$) 1st-order eqs. involving only $\wedge$ and d of p_α-forms (no metric!):

$$R^\alpha := dX^\alpha + Q^\alpha(X) \ \approx \ 0 \ , \qquad Q^\beta \frac{\partial Q^\alpha}{\partial X^\beta} \equiv 0$$

define a *free differentiable algebra* (FDA) ($\Re$,Q), $\Re = \{X^\alpha = X^{\mathbf{p}_\alpha}(x)\}$.

$\Rightarrow$ Gauge invariance of $\{R^\alpha \approx 0\}$: $G^\alpha \ := \ \delta_\epsilon X^\alpha \ = \ d\epsilon^\alpha - \epsilon^\beta \partial_\beta Q^\alpha$

- Gauge symmetry $\forall\ X^{\mathbf{p}_\alpha > 0} \Rightarrow$ all local dof in the 0-forms $X^{\mathbf{0}}$! Non-topological if the 0-form module is ∞-dimensional.

- 1-form sector: $d\Omega + \Omega^2 = 0 \quad \Rightarrow \quad \Omega = \Omega^a T_a \in \mathfrak{g}$, Lie algebra

$$Q^\alpha(X)|_\Omega = f^\alpha{}_{\beta\gamma}\, \Omega^\beta \wedge \Omega^\gamma \qquad \Rightarrow \qquad f^\beta{}_{[\gamma\delta}\, f^\alpha{}_{\eta]\beta} = 0$$

- Linearize around Ω, $X^\alpha = \Omega + \delta X^\alpha$: fluctuation p-form eqs:

$$\mathcal{D}|X^{\mathbf{p}}(x)\rangle = (d + \Omega^a t_a)\,|X^{\mathbf{p}}(x)\rangle \ , \qquad Q^2 = 0 \ \Rightarrow \ [t_a, t_b] = f_{ab}^c\, t_c$$

$\Rightarrow$ Fluctuation p-forms arranged in $\mathfrak{g}$-modules!

HS algebra (totally sym bosonic fields)

HS gauge theories: $\mathfrak{g}$ = (A)dS isometry alg. Manifest sym of free eqs.
= ∞-dim. extension $\mathfrak{ho}$(D-1,2) = $\mathfrak{Lie}[\mathscr{A}]$ = $\mathfrak{Lie}[\mathcal{U}(\mathfrak{so}$(D-1,2)) $/\mathcal{I}(V)] \supset \mathfrak{g}$

$\mathfrak{so}$(D-1,2) : $[M_{AB}, M_{CD}]_\star = 4i\eta_{[C|[B}M_{A]|D]}\ , \quad A = 0', 0, 1, ..., D-1$

With $P_a = \lambda M_{0'a}$, a = 0,1,...,D-1, Lorentz-cov. slicing $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$

$$[M_{ab}, M_{cd}]_\star = 4i\eta_{[c|[b}M_{a]|d]}\ , \quad [M_{ab}, P_c]_\star = 2i\eta_{c[b}P_{a]}\ , \quad [P_a, P_b]_\star = i\lambda^2 M_{ab}$$

$\mathcal{U}(\mathfrak{so}$(D-1,2)) = { totsym products of Ms & Ps }
Factorization of $\mathcal{I}(V)$ leaves traceless two-rows YD:

$$\mathcal{I}[V] = \left\{X = V \star X' \text{ for } X' \in \mathcal{U}\right\}\ , \quad V = l^{AB}V_{AB} + l^{ABCD}V_{ABCD}$$

$$V_{AB} \equiv \frac{1}{2}M_{(A}{}^{C}M_{B)C} - \frac{1}{D+1}\eta_{AB}C_2 \approx 0\ , \quad V_{ABCD} \equiv M_{[AB}M_{CD]} \approx 0$$

$X \in \mathscr{A}$: $$X = \sum_{m\geq n\geq 0} X^{(m,n)}_{a(m),b(n)} M^{a_1b_1}\cdots M^{a_nb_n}P^{a_{n+1}}\cdots P^{a_m}$$

Trace: $\mathrm{Tr}'[X] = X^{(0,0)}$ ⟹ $\langle X|Y\rangle = \mathrm{Tr}'[X^\dagger \star Y]$

Adjoint and Twisted-Adjoint Modules

Antiautomorphism: $\tau(X \star Y) = \tau(Y) \star \tau(X)\ , \quad \tau(M_{AB}) = -M_{AB}$

Automorphism: $\pi(X \star Y) = \pi(X) \star \pi(Y)\ , \quad \pi(M_{ab}) = M_{ab}\ , \ \pi(P_a) = -P_a$

Gauge fields $\in \mathfrak{ho}$(D-1,2) *(master 1-form)*:

$$A(x) = \sum_{s=0}^{\infty} \sum_{t=0}^{s-1} \frac{i}{2} dx^\mu A^{\{s-1,t\}}_{\mu,a_1\dots a_{s-1},b_1\dots b_t}(x) M^{a_1 b_1} \dots M^{a_t b_t} P^{a_{t+1}} \dots P^{a_{s-1}}$$

Gauge invariant curvatures and derivatives: **twisted adj** rep. $\mathcal{T}(\mathfrak{ho}) \ni \Phi$

$$\widetilde{X}\Phi := \mathcal{T}(X)(\Phi) := [X, \Phi]_{\star,\pi} := X \star \Phi - \Phi \star \pi(X)$$ *(master 0-form)*

$$\Phi(x) = \sum_{s,k=0}^{\infty} \frac{1}{k!} \Phi^{\{s+k,s\}}_{a_1\dots a_{s+k},b_1\dots b_s}(x)\, M^{a_1 b_1} \dots M^{a_s b_s} P^{a_{s+1}} \dots P^{a_{s+k}}$$

N.B.: spin-s sector spanned by all {s+k,s}-tensors, k=0,1,2...

(upon constraints, all on-shell-nontrivial covariant derivatives of the physical fields, *i.e.*, all the dynamical information is in the 0-form at a point)

Unfolding → dynamics "dual" to fibre $\mathcal{T}(\mathfrak{ho})$.

(U)IRs of $\mathfrak{so}$(D-1,2)

- Dof of FT unfolded system in $\mathcal{T}(\mathfrak{ho})$ (Lorentz-covariantly sliced) $\Rightarrow$ look for a map $\mathcal{T}$–basis monomials $\leftrightarrow$ massless AdS$_D$ (U)IRs

- Noncompact algebra $\Rightarrow$ ∞-dimensional UIRs
- Compact time translation ($E \sim P_0 \sim M_{0'0}$) $\Rightarrow$ discrete energy spectrum

E induces the splitting: $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$

$$\begin{array}{ccccc} \mathfrak{g}_0 & = & \mathfrak{so}(D-1,2) & \oplus & \mathfrak{so}(2) \\ & & (M_{rs} & , & E) \end{array}$$ compact subalgebra, $\mathfrak{g}_\pm = \{L_r^\pm = M_{0r} \mp iM_{0'r}\}$ ladder ops.

$$[L_r^-, L_s^+] = 2iM_{rs} + 2\delta_{rs}E\ , \quad [E, L_r^\pm] = \pm L_r^\pm\ , \quad [M_{rs}, M_{tu}] = 4i\delta_{[t|[s}M_{r]|u]}$$

l.w. IR $\rightarrow$ $\mathcal{D}(e_0,(s_0))$, built on l.w.s. $|e_0,(s_0)\rangle$:

$L_r^-|e_0,(s_0)\rangle = 0$ $\Rightarrow$ E bounded from below

$$\mathcal{D}(e_0,(s_0)) = \mathcal{V}(e_0,(s_0))/I$$

$$\mathcal{V}(e_0,(s_0)) = \{L_{r_1}^+ \dots L_{r_n}^+|e_0,(s_0)\rangle\}_{n=0}^\infty\ , \quad I = \{\mathcal{V}(e_m,(s')) :\ L_r^-|e_m,(s')\rangle = 0\}$$

Factoring out singular submodules $\Rightarrow$ multiplet shortening.

(U)IRs of $\mathfrak{so}$(D-1,2)

- (Composite) Massless: $e_0 = s_0+2\varepsilon_0 \rightarrow \mathcal{D}(s_0+2\varepsilon_0,(s_0))$ (scalar & shadow $\mathcal{D}(2\varepsilon_0,(0))$ and $\mathcal{D}(2,(0))$)
- Singletons: scalar $\mathcal{D}(\varepsilon_0,(0))$, spinor $\mathcal{D}(\varepsilon_0+1/2,(1/2))$

(+ "anti-particles": $\mathcal{D}^{-}(-e_0,s_0) = \pi(\mathcal{D}(e_0,s_0))$) $[\varepsilon_0 = (D-3)/2]$

Massless particles = two-singletons composites! *(Flato-Fronsdal, '78, Vasiliev '04, Engquist-Sundell '05)*

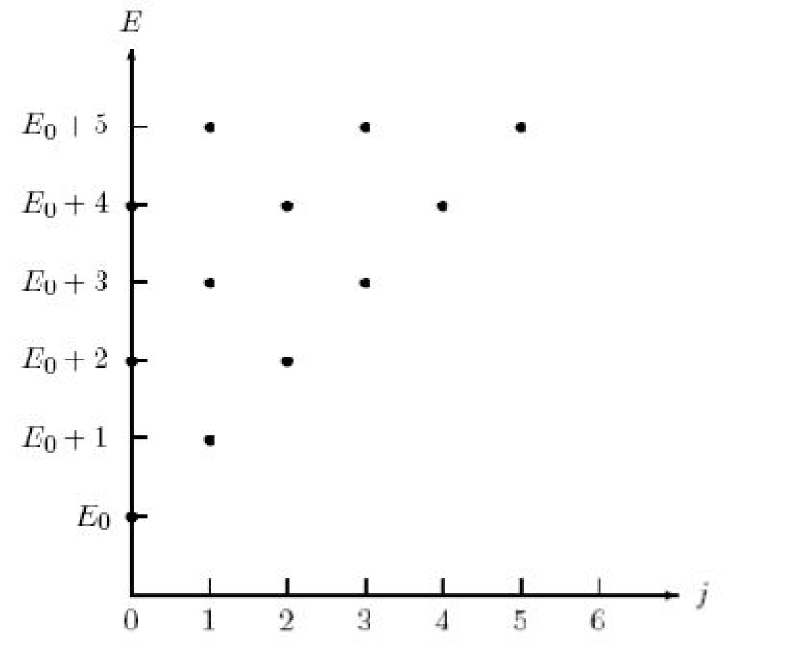
$$\mathcal{D}(\epsilon_0,(0)) \otimes \mathcal{D}(\epsilon_0,(0)) = \bigoplus_{s=0}^{\infty} \mathcal{D}(s+2\epsilon_0,(s)) \ ,$$

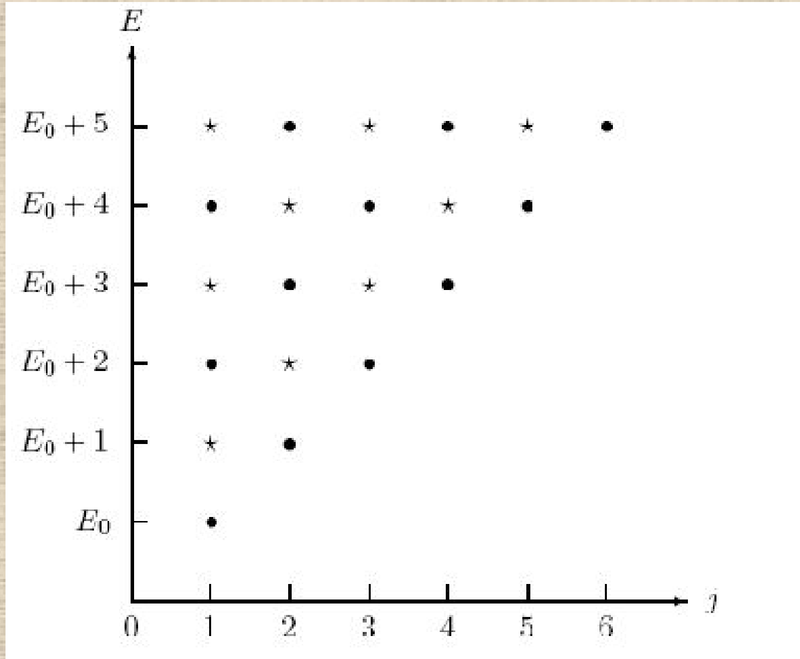
$$D=4: \ \mathcal{D}(1,1/2) \otimes \mathcal{D}(1,1/2) = \mathcal{D}(2,0) \oplus \bigoplus_{s=1}^{\infty} \mathcal{D}(s+1,s)$$

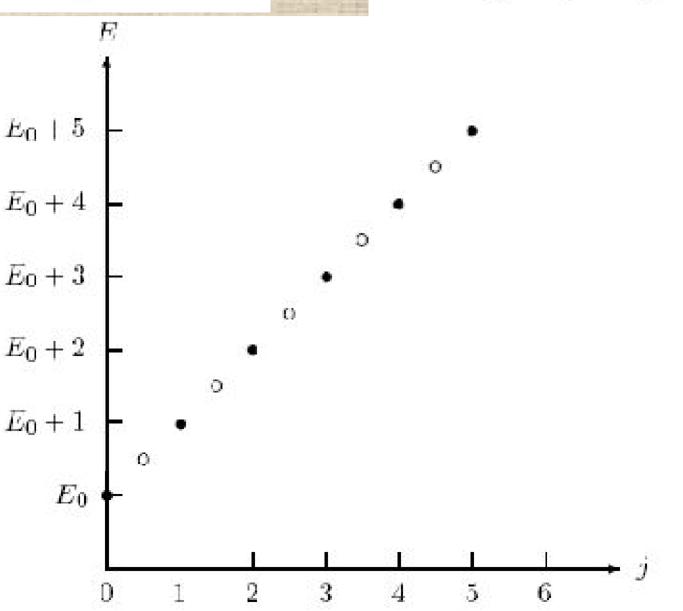
Composite l.w. states:

$$|s+2\epsilon_0,(s)\rangle_{r_1\ldots r_s} = \sum_{k=0}^{s} \alpha_{k,s} (L^+_{\{r_1}\ldots L^+_{r_k})(1)(L^+_{r_{k+1}}\ldots L^+_{r_s\}})(2)|\epsilon_0,0\rangle_1|\epsilon_0,0\rangle_2$$

Weight diagrams







Main Ideas and Results

- To exhibit the correspondence states (U)IRs ↔ twisted-adjoint ops.

$$\mathcal{D}(s+2\epsilon_0;(s)) \quad\longleftrightarrow\quad \mathcal{T}_{(s)} \ni \Phi_{(s)} = \sum_{k=0}^{\infty} \frac{i^k}{k!} \Phi^{a(s+k),b(s)} T_{a(s+k),b(s)} ,$$
$$T_{a(s+k),b(s)} = M_{\{a_1 b_1} \dots M_{a_s b_s} P_{a_{s+1}} \dots P_{a_{s+k}\}}$$

slice $\mathcal{T}$ ($\mathfrak{so}(2) \oplus \mathfrak{so}$(D-1))-covariantly → (inv.) *harmonic expansion*

$$\mathcal{T}\Big|_m \longrightarrow \mathcal{M} := \mathcal{T}\Big|_g = \bigoplus_{s=0}^{\infty} \mathcal{M}_{(s)} , \quad \mathcal{M}_{(s)} = \bigoplus_{\substack{e\,\in\,\mathbf{Z} \\ j_1 \geq s \geq j_2 \geq 0}} \mathbf{C} \otimes T^{(s)}_{e;(j_1,j_2)}$$

and look for lowest (highest)-weight elements $T_{e_0;(s_0)}$.

N.B.: $\tilde{C}_{2n}[\mathcal{T}_{(s)}] = \tilde{C}_{2n}[\mathcal{M}_{(s)}] = C_{2n}[\mathcal{D}(s+2\varepsilon_0,(s))] = C_{2n}[\mathfrak{ho}_{(s)}]$

- Work in $\mathcal{U}[\mathfrak{g}]$: $\mathfrak{g}$-reps. defined by factoring out ideals . *(Duflo,Dixmier,…)*

e.g.: - $\mathcal{J}$[V] = annihilating ideal of scalar singleton, $\mathcal{J}[V] = \mathcal{J}[\mathcal{D}_0]$ (= $\mathcal{J}[\mathcal{D}_{1/2}]$ in D=4)

- Casimirs are fixed in $\mathscr{A}$, $S := \mathscr{A} * \mathbf{X}$, $C_{2n}[S] = C_{2n}[\mathcal{D}_0]$ (=$C_{2n}[\mathcal{D}_{1/2}]$ in D=4)

- Just as $|0\rangle\langle 0| = :e^{-N}:$, one-pt. states = non-polynomial *f(M,P)* (∈ some analytic completion of $\mathcal{U}[\mathfrak{g}]$).

Main Ideas and Results

- No *a priori* l.(h.)w.s $\Rightarrow$ fibre approach is sensitive also to other irreps! (unbounded-E modules).
  $\Rightarrow$ $\mathcal{D}(s+2\varepsilon_0,(s))$, massless one-pt. states, contained in $\mathscr{A}$ as invariant subspaces of indecomposable module $\boxed{\mathcal{M} \;=\; \mathcal{D} \oplus\!\!\!\!\!\!\!\in \mathcal{W}}$
  $\mathcal{W}$ = lowest-spin module containing (linearized) runaway solutions.
  $\Rightarrow$ Prior to imposing b.c., $\mathcal{T}(\mathfrak{ho})$ contains more than one-pt. states.

- The entire $\mathcal{M}$ can be generated via $\mathcal{T}(\mathfrak{ho})$-action from static (even/odd) runaway mode(s) $\phi_{0;(0)}$ (and $\phi_{0;(1)}$) of the free scalar field

D = 4 :

$$ds^2 = \frac{1}{\cos^2\xi}(-dt^2 + d\xi^2 + \sin^2\xi d\Omega^2_{S^2})$$

Static, $\ell = 0$ runaway field	unfolding	Static, $\ell = 0$ analytic $\mathcal{A}$ function
$\phi_{0;(0)}(\xi) \;=\; \frac{\xi}{\tan\xi}$	$\leftrightarrow$	$\frac{\sinh 4E}{4E}$

- $\mathcal{M}$ can be endowed with (rescaled) Tr-norm (proved positive-def. for even scalar l.s. module) and factorizable in terms of *angletons*.

$$\mathcal{S}^{\pm} \;=\; \mathcal{A} \star T^{(0)}_{(\pm)}$$

Compact twisted-adjoint module

- $\mathcal{M}_{(s)}$ spanned by series expansions in $\mathfrak{m}$-cov. elements $T_{a(s+k),b(s)}$:

$$\left[T^{(s)}_{e;(j_1,j_2)}\right]_{r(j_1),t(j_2)} = \sum_{n=0}^{\infty} \underbrace{f^{(s)}_{e;(j_1,j_2);n}} \left[T^{(s)}_{(j_1,j_2);n}\right]_{r(j_1),t(j_2)} \;, \qquad \left[T^{(s)}_{(j_1,j_2);n}\right]_{r(j_1),t(j_2)} = T_{0(n)\{r(j_1),t(j_2)\}0(s-j_2)}$$

generating function $f^{(s)}_{e;(j_1,j_2)}(z) = \sum_{n=0}^{\infty} f^{(s)}_{e;(j_1,j_2);n} z^n$ (*spectral f.*) determined uniquely by

$$\widetilde{E}\, T^{(s)}_{e;(j_1,j_2)} = \{E, T^{(s)}_{e;(j_1,j_2)}\}_\star = e\, T^{(s)}_{e;(j_1,j_2)} \;, \qquad f^{(s)}_{e;(j_1,j_2);0} = 1$$

- $\mathcal{J}[V] = 0$:

$$\widetilde{L}^{\pm}_r \left[T^{(s)}_{e;(s,j_2)}\right]_{rt(s-1),u(j_2)} = 0 \quad \text{for } j_1 = s \geq 1 \text{ and } j_2 < s \,,$$

$$\mathbf{P}_{\{j_1,j_2,1\}} \left[\widetilde{L}^{\pm}_u \left[T^{(s)}_{e;(j_1,j_2)}\right]_{r(j_1),t(j_2)}\right] = 0 \quad \text{for } j_2 \geq 1$$

- $\mathcal{M}_{(s)}$ splits under $\mathcal{U}(\mathcal{T}[\mathfrak{g}])$ in even/odd submodules :

$$\mathcal{M}^{(\pm)} = \bigoplus_{\substack{e;\,(j_1,\,j_2)\\ e+j_1+j_2=\sigma^{\pm} \bmod 2}} \mathbf{C} \otimes T^{(s)}_{e;(j_1,j_2)}$$

$$(\sigma_\pm = (1 \mp 1)/2)$$

Compact twisted-adjoint module

- $\mathcal{M}_{(s)}$ generated via $\mathcal{U}(\mathcal{T}[\mathfrak{g}])$ from elements with $e = 0$ and minimal $j_1 + j_2$: $s=0\ :\quad T^{(0)}_{(\pm)} = T^{(0)}_{0;(\sigma_\pm)}\ ;\qquad s>0\ :\quad T^{(s)}_{(\pm)} = T^{(s)}_{0;(s,\sigma_\pm)}$
  (*static ground states*)
  and all $\mathcal{M}$ from even/odd scalar ground states via $\mathcal{U}(\mathcal{T}[\mathfrak{ho}])\, T_{(\pm)}{}^{(0)}$
  $\Rightarrow$ non-polynomiality included in their spectral functions:

$$f^{(0)}_{0;(0)}(z) = \sum_{p=0}^{\infty} \frac{(4z)^{2p}(\epsilon_0+\frac{3}{2})_{2p}}{(2)_{2p}(2\epsilon_0+1)_{2p}} = {}_2F_3\left(\frac{2\epsilon_0+3}{4}, \frac{2\epsilon_0+5}{4}; \tfrac{3}{2}, \epsilon_0+\tfrac{1}{2}, \epsilon_0+1; 4z^2\right)\ ,$$

$$f^{(0)}_{0;(1)}(z) = \sum_{p=0}^{\infty} \frac{(\epsilon_0+\frac{5}{2})_{2p}\, z^{2p}}{p!(2)_p(\epsilon_0+1)_p(\epsilon_0+2)_p} = {}_2F_3\left(\frac{2\epsilon_0+5}{4}, \frac{2\epsilon_0+7}{4}; 2, \epsilon_0+1, \epsilon_0+2; 4z^2\right)$$

D = 4 : $f^{(0)}_{0;(0)}(z) = \dfrac{\sinh 4z}{4z}\ , \qquad f^{(0)}_{0;(1)}(z) = \dfrac{3}{16z^2}\left(\cosh 4z - \dfrac{\sinh 4z}{4z}\right)$

(Sezgin-Sundell '05)

- Scalar even lowest-spin module: $\mathcal{W}^{(+)}_{(0)} = \bigoplus_{|e|\leq j} \mathbf{C} \otimes T^{(0)}_{e;(j)}$

Lowest-weight submodules

- L.w. states in $\mathcal{M}_{(s)}$ are solutions of:

$$\widetilde{L}_r^- T^{(s)}_{e;(j_1,j_2)} \;=\; L_r^- \star T^{(s)}_{e;(j_1,j_2)} - T^{(s)}_{e;(j_1,j_2)} \star L_r^+ \;=\; 0$$

- Equating Casimir ops. for l.w.s. and $\mathcal{T}_{(s)}$ and using ideal relations $\Rightarrow$ l.w. admissibility conditions:

$$j_2 \;=\; 0: \qquad \underbrace{j_1 \;=\; s\;, \quad e \;=\; s+2\epsilon_0}_{\mathcal{D}(s+2\varepsilon_0,(s))} \qquad \text{and} \qquad \underbrace{j_1 \;=\; s \;=\; 0\;, \quad e \;=\; 2}_{\mathcal{D}(2,(0))}$$

$$j_2 \;=\; s \geq 1: \qquad j_1 \;=\; j_2 \;=\; s \;=\; 1\;, \quad e \;=\; 2 \qquad \mathcal{D}(2,(s,s))$$

- s = 0 : $T^{(0)}_{2\epsilon_0;(0)} \;=\; {}_1F_1(\epsilon_0+\frac{3}{2};2;-4E)\;, \qquad T^{(0)}_{2;(0)} \;=\; {}_1F_1(\epsilon_0+\frac{3}{2};2\epsilon_0;-4E)$

D = 4 : $\qquad T^{(0)}_{1;(0)} \;=\; e^{-4E}\;, \qquad T^{(0)}_{2;(0)} \;=\; (1-4E)e^{-4E}$

- The Verma module built on top of l.w. is an invariant submodule of $\mathcal{M}_{(s)}$ (indecomposable structure changes with dimension).

Lowest-weight submodules

- Similarly for s > 0, where the l.w. elements are (similar for T_2 in D=4)

$$\left[T^{(s)}_{s+2\epsilon_0;(s)}\right]_{r(s)} = \sum_{k=0}^{s}(-1)^{s-k}\alpha_{s;k}L^+_{\{r_1}\star\cdots\star L^+_{r_k}\star T^{(0)}_{2\epsilon_0;(0)}\star L^-_{r_{k+1}}\star\cdots\star L^-_{r_s\}}$$

- Two-sided, enveloping-alg. version of Flato-Fronsdal!, with

$$T^{(0)}_{2\epsilon_0;(0)} \simeq |\epsilon_0;(0)\rangle\langle\epsilon_0;(0)|$$

(can be mapped to one-sided version in a mathematically precise way using a reflector state.)

→ composite nature of compact twisted-adjoint l.w. elements.

- Can be verified by studying properties of compact scalar elements. Ideal relations imply:

$$\begin{aligned} E\star T^{(0)}_{e;(0)} &= T^{(0)}_{e;(0)}\star E = \frac{e}{2}T^{(0)}_{e;(0)}\ , \\ L^-_r\star T^{(0)}_{e;(0)} &= 0 = T^{(0)}_{e;(0)}\star L^+_r \ \text{ only if } \ e = 2\epsilon_0, 2 \\ M_{rs}\star T^{(0)}_{e;(0)} &= 0 \ \text{ only if } \ e = \pm 2\epsilon_0 \end{aligned}$$

⇒ one-sidedly $\mathcal{S}^{(0)}_{2\epsilon_0;(0)} := \mathcal{A}\star T^{(0)}_{2\epsilon_0;(0)} \simeq \mathcal{D}(\epsilon_0;(0))$

and two-sidedly $\mathcal{D}^+ := \mathcal{A}\star T^{(0)}_{2\epsilon_0;(0)}\star\mathcal{A} \simeq \mathcal{D}_0\otimes\mathcal{D}_0^*$

(related works: Shaynkman, Vasiliev '01 Vasiliev '02)

Conclusions

- Reflection of dual modules gives composite-state presentation (reflector state does the job) on l.w. subsectors of twisted-adjoint.
- It maps explicitly twisted-adjoint ops. to its compact-state content. (but factorization and explicit reflection only in composite-massless sectors)
- Opposite mapping can be performed, i.e., assembling compact states into Lorentz-covariant ones (harmonic expansion) and reflecting.
- Applied to Adjoint representation (non-unitary):

$$R_2 : \ \delta|A\rangle = [\epsilon(1) + \pi(\epsilon(2))] \star |A\rangle \ \rightarrow \ \delta A = [\epsilon, A]_\star$$

equivalent to standard left action $\varepsilon(1) + \varepsilon(2)$ on the non-unitary Singleton $\otimes$ Anti-Singleton module!

FF-like decomposition:

$$\boxed{(\mathcal{D}_0^+ \otimes \mathcal{D}_0^-) \oplus (\mathcal{D}_0^- \otimes \mathcal{D}_0^+) \ = \ \bigoplus_{s=0}^{\infty} \mathcal{D}(-(s-1); (s-1))}$$

Conclusions & Outlook

- Fibre/enveloping algebra approach is natural in unfolding. Insight on nature of field-th. representations (twisted-adj. content prior to b.c., compact-space meaning of Chevalley-Eilenberg cocycles…) and useful to rep.theory (independent of oscillator realization, analysis of dS irreps in Lorentz-cov. presentation…).

- Fibre harmonic expansion generalized to analyse content of twisted adjoint rep. and unitarity for mixed-symmetry fields in AdS.
(Boulanger, C.I., Sundell '08)

- What is the analog of the singleton annihilating ideal for mixed-sym fields ?

- Interesting possible generalizations also involving massive and partially massless fields (generalizing fibre analysis to affine extensions of HS algebra).

Compact twisted-adjoint module

Admissibility criterion: spectrum of phys. fields matches doubletons

(*Konstein-Vasiliev, '89*)

Now: **map** doubletons (left module) to HS Master Fields (double-sided module)

- From compact to Lorentz-covariant basis of states
- Reflecting a LL into a LR-module, preserving rep. properties

$$D_0^{\otimes 2} \oplus D_{1/2}^{\otimes 2} \longrightarrow |\Phi\rangle = \sum_{m,n} \phi_{m,n} |m\rangle_1 |n\rangle_2 \xrightarrow{R_2} \Phi(M_{ab}, P_a)$$

- s=0: find a Lorentz-scalar superposition $|\mathbf{1}\rangle_0 = \psi(x)|1,0\rangle \in (D_0)^{\otimes 2}$:

$$x \equiv L_r^+ L_r^+ = y^2$$

$$M_{ab}|\mathbb{1}\rangle_0 = 0 \ , \quad i.e. \ M_{0r}\psi(x)|1,0\rangle = 0$$

a harmonic eq. in y $\Rightarrow$ $\quad |\mathbb{1}\rangle_0 = \cos(y)|1,0\rangle \ \in Env(so(3,2))$

Degeneracy! Also possible to expand on states in $D(2,0) \in (D_{1/2})^{\otimes 2}$.
Same procedure yields $\quad |\mathbb{1}\rangle_{1/2} = \frac{\sin(y)}{y}|2,0\rangle \ \in Env(so(3,2))$

Mapping Doubletons to Master Fields

Oscillator realization: $|\mathbb{1}\rangle_{1/2} = \sin y|1,0\rangle \;\Rightarrow\; |\mathbb{1}\rangle_{0+i(1/2)} = e^{iy}|1,0\rangle$

$$|1/2,0\rangle\langle 1/2,0| =: \; e^{-a^{\dagger\, i}a_i} \; :$$

Define Reflector: $R(|1/2,0\rangle) = \langle 1/2,0| \;, \quad R(a^{\dagger\, i}) = ia^i \;, \quad R(f\star g) = R(g)\star R(f)$

$$\Rightarrow \quad R_2(e^{iy}|1/2,0\rangle_1|1/2,0\rangle_2) =: \; e^{a^{\dagger\, i}a_i}|1/2,0\rangle\langle 1/2,0| \;:= \mathbb{1}$$

i.e., the Lorentz-scalar in Φ!

R gives **correct (tw. Adj.) tranformations**!

$$R_2 \quad : \quad \delta|\Phi\rangle = [\epsilon(1)+\epsilon(2)]\star|\Phi\rangle \;\longrightarrow\; \delta\Phi = \epsilon\star\Phi - \Phi\star\pi(\epsilon)$$

(since $R(\epsilon|n\rangle) = -\langle n^c|\,\pi(\epsilon)$)

By HS-symmetry, this extends to all {s+k,s}-monomials in tw. Adj.!

- General result: L-basis: $|M^s P^k\rangle \sim$ **$e^{iy} \times \mathrm{Pol}(a^i, a^{\dagger i})$** $|1/2,0\rangle_1|1/2,0\rangle_2$
  Reflection : $R_2(\mathrm{Pol}(a^i,a^{\dagger i})) = M^s P^k$

More on the Reflector

- Map can be performed in abstract algebra and in D dimensions! Intro the REFLECTOR $|\mathbf{1}\rangle_{12}$ s.t.

$$(M_{ab}(1)+M_{ab}(2))|\mathbb{1}\rangle_{12}=0\ ,\quad (P_a(1)-P_a(2))|\mathbb{1}\rangle_{12}\ =\ 0$$

$$M^sP^k = M^sP^k \star \mathbb{1} \xrightarrow{R_2^{-1}} \boxed{|M^sP^k\rangle_{12}\ =\ (M^sP^k)(1)|\mathbb{1}\rangle_{12}}$$

Exp-states "special" only because normalizable in a certain inner product ($\leftrightarrow$ STr in twisted Adj.)

- **Inverse map**: $\Phi(M_{ab},P_a)\ \longrightarrow\ \phi_{e_0,s_0}=\phi_{e_0,s_0}(M_{rs},E,L_r^{\pm})\ \xrightarrow{R_2^{-1}} D_0^{\otimes 2}\oplus D_{1/2}^{\otimes 2}$

1. Single out l.w. combination of ops. (with definite e_0 and s_0)
2. Inverse reflection to doubleton states

Scalar: $[M_{rs},\phi_{e_0,s_0}]_\pi\ =\ 0\ ,\quad [E,\phi_{e_0,s_0}]_\pi\ =\ e_0\ ,\quad [L_r^-,\phi_{e_0,s_0}]_\pi\ =\ 0$

2 solutions: $\phi_{1,0}\ =\ \exp(-4E)\ ,\qquad \phi_{2,0}\ =\ (1-4E)\exp(-4E)$

Conclusions

- Reflection map connects very different descriptions:
  a) L.w. modules → global bkgrd properties (finite-E fluct.)
  b) Tw.-Adjoint basis → no b.c., only local data
  (contains scalars with N&D b.c.; in D=4 each spin-s sector is furtherly decomposed in (anti)-selfdual;...)

- Also other nonpolynomial objects in Φ → states outside l.w. modules!
  ⇒ **Full** twisted adjoint (indecomposable) spin-0 module is

$$\boxed{\mathcal{M}_0 \;=\; \mathcal{W}_0 \inplus D(1,0) \oplus D(2,0) \oplus \widetilde{D}(1,0) \oplus \widetilde{D}(2,0)}$$

with ground states

$$\phi_{0,0} \;=\; \frac{\sinh 4E}{4E}\;, \qquad (\phi_{0,1})_r \;=\; P_r \sum_n \frac{(4E^2)^n}{n!(5/2)_n}$$

Conclusions & Outlook

- Adjoint ~ nonunitary, unbounded-E l.w. realization,

$$R_2 : \ \delta|A\rangle = [\epsilon(1) + \pi(\epsilon(2))] \star |A\rangle \ \rightarrow \ \delta A = [\epsilon, A]_\star$$

equivalent to standard left action ε(1)+ ε(2) on the nonunitary module Singleton ⊗ Anti-Singleton !

FF-like decomposition: $D(1/2,0) \otimes \widetilde{D}(-1/2,0) \ \sim \ \sum_s \mathcal{V}_s$

- Extension to O(D+1;C), *i.e.* arbitrary signature
  (interesting **exact** solution in different signatures → *C.I., E.Sezgin, P.Sundell, arXiv 0706.2983 [hep-th])*

- Usata per mixsym. Possibly interesting extension to massive HS & partially massless!

In components

$$|\{s+k,s\};\{s+t,j\}\rangle = e^{iy} \times \mathrm{Ply}_{s,t,j}(a^i, a^{\dagger\, i})|1/2,0\rangle_1 |1/2,0\rangle_2$$

$$\xrightarrow{R_2} R_2(\mathrm{Ply}_{s,t,j}(a^i, a^{\dagger\, i})) = \begin{cases} M_{0r_1}...M_{0r_s} P_{r_{s+1}}...P_{r_{s+t}}(P_0)^{s+k-t} \;, \; j=0 \\ M_{qr_1} M_{0r_2}...M_{0r_s} P_{r_{s+1}}...P_{r_{s+t}}(P_0)^{s+k-t} \;, \; j=1 \end{cases}$$

(For general {s+k,s}: 1) decompose 4d to 3d YD, $|\{s+k,s\}\rangle \rightarrow$
$|\{s+k,s\};\{s+t,0\}\rangle$, $|\{s+k,s\};\{s+t,1\}\rangle$, t=0,...,k ($M_{0r}$~ step op.)
2) k=0 → bottom/top superpositions ~ trigonometric ψ(y) on lws $|s+1,s\rangle$;
k>0 → descendants of k=0 via left-action of P^k)

The Vasiliev Equations

NC extension, x →(x,Z): $[z_\alpha, z_\beta]_\star = -2i\varepsilon_{\alpha\beta}\ , \quad [\bar z_{\dot\alpha}, \bar z_{\dot\beta}]_\star = -2i\varepsilon_{\dot\alpha\dot\beta}$

$$d \to \hat d = d + d_Z$$

$$A(x|Y) \to \hat A(x|Z,Y) \equiv (dx^\mu \hat A_\mu + dz^\alpha \hat A_\alpha + d\bar z^{\dot\alpha} \hat A_{\dot\alpha})(x|Z,Y)\ , \quad A_\mu(x|Y) = \hat A_\mu|_{Z=0}$$

$$\Phi(x|Y) \to \hat\Phi(x|Z,Y)\ , \quad \Phi(x|Y) = \hat\Phi(x|Z,Y)|_{Z=0}$$

$$\boxed{\begin{aligned} &\hat F \equiv \hat d \hat A + \hat A \star \hat A = \frac{i}{4}(dz^\alpha \wedge dz_\alpha \hat\Phi \star \kappa + d\bar z^{\dot\alpha} \wedge d\bar z_{\dot\alpha} \hat\Phi \star \bar\kappa) \\ &\hat{\mathcal D}\hat\Phi(x|Y,Z) \equiv \hat d\hat\Phi + \hat A \star \hat\Phi - \hat\Phi \star \bar\pi(\hat A) = 0 \end{aligned}}$$

Local sym: $\delta\hat A = \hat{\mathcal D}\hat\epsilon\ , \quad \delta\hat\Phi = -[\hat\epsilon, \hat\Phi]_\pi$

$$\hat F_{\mu\nu} = \hat F_{\alpha\mu} = \hat F_{\dot\alpha\mu} = \hat F_{\alpha\dot\alpha} = 0\ ,$$
$$\hat F_{\alpha\beta} = -\frac{i}{2}\epsilon_{\alpha\beta}\hat\Phi \star \kappa\ ,$$
$$\hat F_{\dot\alpha\dot\beta} = -\frac{i}{2}\epsilon_{\dot\alpha\dot\beta}\hat\Phi \star \bar\kappa\ ,$$
$$\hat{\mathcal D}_\mu\hat\Phi = \hat{\mathcal D}_\alpha\hat\Phi = \hat{\mathcal D}_{\dot\alpha}\hat\Phi = 0$$

Solving for Z-dependence yields consistent nonlinear corrections as an expansion in Φ.

For space-time components, projecting on phys. space {Z=0} → $\boxed{\hat F_{\mu\nu}(x|A,\Phi;Y)|_{Z=0} = 0\ , \quad (\hat{\mathcal D}_\mu\hat\Phi)(x|\Phi;Y)|_{Z=0} = 0}$

Appendix II

Also the other way around! (base ↔ fiber evolution)

Locally give x-dep. via gauge functions (space-time ~ pure gauge!)...

$$\widehat{A}_\mu = \widehat{L}^{-1}\star\partial_\mu\widehat{L}\ , \quad \widehat{A}_\alpha = \widehat{L}^{-1}\star(\widehat{A}'_\alpha + \partial_\alpha)\star\widehat{L}\ , \quad \widehat{\Phi} = \widehat{L}^{-1}\star\widehat{\Phi}'\star\pi(\widehat{L})$$

$$\widehat{L} = \widehat{L}(x|Z,Y)\ , \quad \widehat{A}'_\alpha = \widehat{A}_\alpha(0|Z,Y)\ , \ \widehat{\Phi}' = \widehat{\Phi}(0|Z,Y)$$

...and substitute in Z-eq.[ns]: $\widehat{F}'_{\alpha\beta} = -\frac{i}{2}\epsilon_{\alpha\beta}\widehat{\Phi}'\star\kappa\ , \ \widehat{F}'_{\alpha\dot{\beta}} = 0\ , \ \widehat{D}'_\alpha\widehat{\Phi}' = 0$

(fiber evolution)

Exact solution can be obtained with: *(Sezgin, Sundell – '05)*

1. $A = L^{-1}\star dL \to AdS_4\ , \quad ds^2_{(0)} = \frac{4dx^2}{(1-x^2)^2}\ , \ (x^2 \leq 1)$

2. SO(3,1)-invariance:

$$[\widehat{M}'_{\alpha\beta}, \widehat{\Phi}']_\pi = 0\ , \quad [\widehat{M}'_{\alpha\beta}, \widehat{A}'_\alpha] = 0 \Rightarrow \begin{array}{rcl} \widehat{\Phi}' & = & f(u,\bar{u})\ , \quad u \equiv y^\alpha z_\alpha \\ \widehat{A}'_\alpha & = & z_\alpha A(u,\bar{u}) \end{array}$$