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Towards the exact spectrum of the $AdS_5 \times S^5$ superstring. II

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with Gleb Arutyunov: arXiv:0710.1568, 0901.1417, 0903.0141

with Ryo Suzuki: work in progress

Exact spectrum of

$AdS_5 \times S^5$ superstring

and

$\mathcal{N} = 4$ SYM

is

NOT

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This year there has been important progress towards finding a solution of this problem

In my talk I'll discuss what have been done and what remains to be understood (a lot!)

String hypothesis for the mirror model

The Bethe-Yang equations for the mirror theory

Arutyunov, Frolov '07, '09(a)

$$
\begin{aligned}
1 &= e^{i\widetilde{p}_k R} \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{I}}} S_{\mathfrak{sl}(2)}^{Q_k Q_l}(x_k, x_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \\
-1 &= \prod_{l=1}^{K^{\mathrm{I}}} \frac{y_k^{(\alpha)} - x_l^-}{y_k^{(\alpha)} - x_l^+} \sqrt{\frac{x_l^+}{x_l^-}} \prod_{l=1}^{K^{\mathrm{III}}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}} \\
1 &= \prod_{l=1}^{K^{\mathrm{II}}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}} \prod_{\substack{l=1\\ l\neq k}}^{K^{\mathrm{III}}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}} .
\end{aligned}
$$

- Here $\widetilde{p}_k$ is the real momentum of a physical Q-particle
- K^{I}, $K^{\mathrm{II}}_{(\alpha)}$ and $K^{\mathrm{III}}_{(\alpha)}$ are the numbers of Q-particles, and auxiliary roots $y_k^{(\alpha)}$ and $w_k^{(\alpha)}$, and $\alpha = 1, 2$. The parameters v are related to y as $v = y + \frac{1}{y}$.
- $S_{\mathfrak{sl}(2)}^{Q_k Q_l}(x_k, x_l)$ is the S-matrix in the $\mathfrak{sl}(2)$ sector of the mirror theory which describes the scattering of a Q_k-particle and a Q_l-particle with momenta $\widetilde{p}_k$ and $\widetilde{p}_l$

String hypothesis

In thermodynamic limit $R, K^{\mathrm{I}}, K^{\mathrm{II}}_{(\alpha)}, K^{\mathrm{III}}_{(\alpha)} \to \infty$ with K^{I}/R and so on fixed solutions of BYE are composed of four different classes of Bethe strings

1. A single Q-particle with real momentum $\widetilde{p}_k$ or, equivalently, rapidity u_k
2. A single $y^{(\alpha)}$-particle (an auxiliary root $y^{(\alpha)}$) with $|y^{(\alpha)}| = 1$
3. $2M$ roots $y^{(\alpha)}$ and M roots $w^{(\alpha)}$ combining into a single $M|vw^{(\alpha)}$-string

$$v_j^{(\alpha)} = v^{(\alpha)} + (M+2-2j)\frac{i}{g}\,, \quad v_{-j}^{(\alpha)} = v^{(\alpha)} - (M+2-2j)\frac{i}{g}\,, \quad j = 1, \dots, M\,,$$
$$w_j^{(\alpha)} = v^{(\alpha)} + (M+1-2j)\frac{i}{g}\,, \quad j = 1, \dots, M\,, \quad v \in \mathbf{R}\,.$$

4. N roots $w^{(\alpha)}$ combining into a single $N|w^{(\alpha)}$-string

$$w_j^{(\alpha)} = w^{(\alpha)} + \frac{i}{g}(N+1-2j)\,, \quad j = 1, \dots, N\,, \quad w \in \mathbf{R}\,.$$

This includes $N = 1$ which has a single real root $w^{(\alpha)}$.

Thermodynamic limit

Introduce densities $\rho(u)$ of particles, and $\bar{\rho}(u)$ of holes; $u \in \mathbf{R}$, $\alpha = 1, 2$.

1. $\rho_Q(u)$ of Q-particles, $-\infty \le u \le \infty$, $Q = 1, \ldots, \infty$
2. $\rho_{y^-}^{(\alpha)}(u)$ of y-particles with Im$(y) < 0$, $-2 \le u \le 2$. The y-coordinate is expressed in terms of u as $y = x(u)$

   $$x(u) = \frac{1}{2}\left(u - i\sqrt{4-u^2}\right) , \quad \mathrm{Im}(x(u)) < 0 \ \text{ for any } u \in \mathbb{C} \,,$$

   the cuts in the u-plane run from $\pm\infty$ to ± 2 along the real lines.
3. $\rho_{y^+}^{(\alpha)}(u)$ of y-particles with Im$(y) > 0$, $-2 \le u \le 2$. The y-coordinate is expressed in terms of u as $y = \frac{1}{x(u)}$
4. $\rho_{M|vw}^{(\alpha)}(u)$ of $M|vw$-strings, $-\infty \le u \le \infty$, $M = 1, \ldots, \infty$
5. $\rho_{N|w}^{(\alpha)}(u)$ of $N|w$-strings, $-\infty \le u \le \infty$, $N = 1, \ldots, \infty$,

and the corresponding densities of holes.

Thermodynamic limit

Let i, j, k run over all the densities. Integral eqs in the thermodynamic limit

$$\rho_i(u) + \bar{\rho}_i(u) = \frac{R}{2\pi}\frac{d\widetilde{p}_i}{du} + K_{ij} \star \rho_j(u)$$

where $\widetilde{p}_i$ does not vanish only for Q-particles.

Left action of $K's$ on ρ_j (the star product) is defined as

$$K_{ij} \star \rho_j(u) = \int \mathrm{d}u' \, K_{ij}(u, u')\rho_j(u')$$

Integration is taken over the corresponding range of u.

$K's$ are expressed through the corresponding S-matrices as

$$K_{ij}(u, v) = \frac{1}{2\pi i}\frac{d}{du}\log S_{ij}(u, v)$$

We will need the right action which is defined as

$$\rho_j \star K_{ji}(u) = \int \mathrm{d}u' \, \rho_j(u')K_{ji}(u', u)\,.$$

Free energy and equations for pseudo-energies

Integral eqs for minimum of free energy per unit length for mirror theory at temperature $T = \frac{1}{L}$ $\Longrightarrow$ The ground state energy of l.c. string theory on the cylinder with circumference $L = P_+$

Light-cone string theory has two different sectors

- Even winding number string states and periodic fermions $\Longrightarrow$ ground state energy is determined by Witten's index of the mirror theory.

  The ground state is BPS $\Longrightarrow$ no quantum corrections to its energy

- Anti-periodic fermions and non-BPS ground state

Free energy and equations for pseudo-energies

To describe both sectors, we consider generalized free energy

$$\mathcal{F}_\gamma(L) = \mathcal{E} - \frac{1}{L} S + \frac{i\gamma}{L}(N_F^{(1)} - N_F^{(2)})\,,$$

- $\mathcal{E}$ is the energy per unit length carried by Q-particles

$$\mathcal{E} = \int \mathrm{d}u \sum_{Q=1}^{\infty} \widetilde{\mathcal{E}}^Q(u)\rho_Q(u)\,, \quad \widetilde{\mathcal{E}}^Q(u) \text{ is } Q\text{-particle energy}$$

- S is the total entropy
- $i\gamma/L$ plays the role of a chemical potential
- $N_F^{(\alpha)}$ is the fermion number which counts the number of $y^{(\alpha)}$-particles

$$N_F^{(1)} - N_F^{(2)} = \int \mathrm{d}u\,(\rho_{y^-}^{(1)}(u) + \rho_{y^+}^{(1)}(u) - \rho_{y^-}^{(2)}(u) - \rho_{y^+}^{(2)}(u))$$

- Minus sign between $N_F^{(1)}$ and $N_F^{(2)}$ is needed for the reality of $\mathcal{F}_\gamma(L)$
- $\gamma = \pi \implies$ Witten's index. $\gamma = 0 \implies$ the usual free energy.

Free energy and equations for pseudo-energies

Free energy: $\mathcal{F}_\gamma(L) = \int \mathrm{d}u \, \sum_k \left[\widetilde{\mathcal{E}}_k \, \rho_k - \frac{i\gamma_k}{L} \, \rho_k - \frac{1}{L} \mathfrak{s}(\rho_k) \right] \,,$

Variations of the densities of particles and holes are subject to

$$\delta\rho_k(u) + \delta\bar{\rho}_k(u) = K_{kj} \star \delta\rho_j \,.$$

Using the extremum condition $\delta\mathcal{F}_\gamma(L) = 0$, one derives the TBA eqs

$$\epsilon_k = L \, \widetilde{\mathcal{E}}_k - \log\left(1 + e^{i\gamma_j - \epsilon_j}\right) \star K_{jk} \,,$$

where the pseudo-energies ϵ_k are $e^{i\gamma_k - \epsilon_k} = \frac{\rho_k}{\bar{\rho}_k} \,,$

and the right action of K_{jk} is used: $\rho_j \star K_{ji}(u) = \int \mathrm{d}u' \, \rho_j(u') K_{ji}(u', u)$

At the extremum $\mathcal{F}_\gamma(L) = -\frac{R}{L} \int \mathrm{d}u \, \sum_k \frac{1}{2\pi} \frac{d\widetilde{p}_k}{du} \log\left(1 + e^{i\gamma_k - \epsilon_k}\right)$

Finally, one gets the energy of the ground state of the l.c. string theory

$$E_\gamma(L) = \lim_{R\to\infty} \frac{L}{R} \mathcal{F}_\gamma(L) = -\int \mathrm{d}u \sum_{Q=1}^{\infty} \frac{1}{2\pi} \frac{d\widetilde{p}^Q}{du} \log\left(1 + e^{-\epsilon_Q}\right)$$

TBA equations

Arutyunov, Frolov '09(b)

- Q-particles $(\,\gamma = \pi + h\,,\; h_\alpha = (-1)^\alpha h\,)$

$$\epsilon_Q = L\,\widetilde{\mathcal{E}}_Q - \log\left(1+e^{-\epsilon_{Q'}}\right)\star K^{Q'Q}_{\mathfrak{sl}(2)} - \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|vw}}\right)\star K^{M'Q}_{vwx}$$

$$- \log\left(1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^-}}\right)\star K^{yQ}_{-} - \log\left(1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^+}}\right)\star K^{yQ}_{+}$$

- y-particles

$$\epsilon^{(\alpha)}_{y^\pm} = -\log\left(1+e^{-\epsilon_Q}\right)\star K^{Qy}_{\pm} + \log\frac{1+e^{-\epsilon^{(\alpha)}_{M|vw}}}{1+e^{-\epsilon^{(\alpha)}_{M|w}}}\star K_M$$

- $M|vw$-strings

$$\epsilon^{(\alpha)}_{M|vw} = -\log\left(1+e^{-\epsilon_{Q'}}\right)\star K^{Q'M}_{xv}$$

$$+\log\left(1+e^{-\epsilon^{(\alpha)}_{M'|vw}}\right)\star K_{M'M} - \log\frac{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^+}}}{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^-}}}\star K_M$$

- $M|w$-strings

$$\epsilon^{(\alpha)}_{M|w} = \log\left(1+e^{-\epsilon^{(\alpha)}_{M'|w}}\right)\star K_{M'M} - \log\frac{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^+}}}{1-e^{ih_\alpha-\epsilon^{(\alpha)}_{y^-}}}\star K_M$$

See also

Bombardelli, Fioravanti, Tateo '09; Gromov, Kazakov,Kozak, Vieira '09

Simplifying the TBA equations

We introduce the Y-functions

$$Y_Q = e^{-\epsilon_Q}\,, \quad Y^{(\alpha)}_{M|vw} = e^{\epsilon^{(\alpha)}_{M|vw}}\,, \quad Y^{(\alpha)}_{M|w} = e^{\epsilon^{(\alpha)}_{M|w}}\,, \quad Y^{(\alpha)}_{\pm} = e^{\epsilon^{(\alpha)}_{y^\pm}}\,,$$

and use the universal kernel

$$(K+1)^{-1}_{MN} = \delta_{MN} - s\left(\delta_{M+1,N} + \delta_{M-1,N}\right)\,, \quad s(u) = \frac{g}{4\cosh\frac{g\pi u}{2}}\,,$$

inverse to $K_{NQ} + \delta_{NQ}$: $\sum_{N=1}^{\infty} (K+1)^{-1}_{MN} \star (K_{NQ} + \delta_{NQ}) = \delta_{MQ}\,,$

where $K_{MN}(u) = K_{M+N}(u) + K_{N-M}(u) + 2\sum_{j=1}^{M-1} K_{N-M+2j}(u)\,,$

$$K_M(u) = \frac{1}{2\pi i}\frac{d}{du}\log\left(\frac{u - i\frac{M}{g}}{u + i\frac{M}{g}}\right) = \frac{1}{\pi}\,\frac{gM}{M^2 + g^2u^2}\,, \quad -\infty \le M \le \infty\,.$$

We often use the following identity

$$\sum_{N=1}^{\infty} (K+1)^{-1}_{MN} \star K_N = s\,\delta_{M1}\,.$$

TBA and Y-equations for w-strings

$$\log Y^{(\alpha)}_{M|w} = \log\left(1+\frac{1}{Y^{(\alpha)}_{M'|w}}\right)\star K_{M'M} - \log\frac{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_-}}{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_+}}\star K_M$$

We apply the inverse kernel, and get the following equation

$$\log Y^{(\alpha)}_{M|w} = I_{MN}\log(1+Y^{(\alpha)}_{N|w})\star s + \delta_{M1}\,\log\frac{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_-}}{1-\frac{e^{ih_\alpha}}{Y^{(\alpha)}_+}}\star s\,,$$

where I_{MN} is the incidence matrix

$$I_{MN} = \delta_{M+1,N} + \delta_{M-1,N}\,.$$

Since the functions $Y^\alpha_\pm$ are defined on the interval $-2 < u < 2$, the integral in the last term is taken from -2 to 2.

Y-system ???

Gromov, Kazakov, Vieira '09

TBA and Y-equations for w-strings

$$\log Y_{M|w}^{(\alpha)} = I_{MN} \log(1 + Y_{N|w}^{(\alpha)}) \star s + \delta_{M1} \log \frac{1 - \frac{e^{ih\alpha}}{Y_-^{(\alpha)}}}{1 - \frac{e^{ih\alpha}}{Y_+^{(\alpha)}}} \star s \,,$$

Define $(f \star s^{-1})(u) = \lim_{\epsilon\to 0^+} \left[f(u + \frac{i}{g} - i\epsilon) + f(u - \frac{i}{g} + i\epsilon) \right]$,

It satisfies the identity $(s \star s^{-1})(u) = \delta(u)\,.$ In general $f \star s^{-1} \star s \neq f\,.$

Introduce the notation $Y_{M|w}^{(\alpha)\pm}(u) \equiv Y_{M|w}^{(\alpha)}(u \pm \frac{i}{g} \mp i0)$, and get the Y-equations

$$\begin{aligned} Y_{M|w}^{(\alpha)+} Y_{M|w}^{(\alpha)-} &= \left(1 + Y_{M-1|w}^{(\alpha)}\right)\left(1 + Y_{M+1|w}^{(\alpha)}\right) \quad \text{if} \ \ M \geq 2\,, \\ Y_{1|w}^{(\alpha)+} Y_{1|w}^{(\alpha)-} &= \left(1 + Y_{2|w}^{(\alpha)}\right) \frac{1 - \frac{e^{ih\alpha}}{Y_-^{(\alpha)}}}{1 - \frac{e^{ih\alpha}}{Y_+^{(\alpha)}}}\,, \qquad |u| \leq 2\,, \\ Y_{1|w}^{(\alpha)+} Y_{1|w}^{(\alpha)-} &= 1 + Y_{2|w}^{(\alpha)}\,, \qquad |u| > 2\,. \end{aligned}$$

Y-system does NOT work for $|u| > 2$

Ground state energy: any L, small h

Naively, for $h=0$ the TBA equations are solved by

work in progress with Ryo Suzuki

$$Y_Q = 0, \qquad Y_+^{(\alpha)} = Y_-^{(\alpha)} = 1, \qquad Y_{M|vw}^{(\alpha)} = Y_{M|w}^{(\alpha)} \neq 0, \qquad e^{ih_\alpha} = 1\,.$$

A subtle point is that the TBA equation for Q-particles is singular at $Y_Q = 0$

$$-\log Y_Q = L\,\widetilde{\mathcal{E}}_Q - \log\left(1 + Y_{Q'}\right) \star K_{\mathfrak{sl}(2)}^{Q'Q} - \log\left(1 + \frac{1}{Y_{M|vw}^{(\alpha)}}\right) \star K_{vwx}^{MQ}$$

$$-\frac{1}{2}\log\frac{1 - \frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}}{1 - \frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}} \star K_Q - \frac{1}{2}\log\left(1 - \frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}\right)\left(1 - \frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}\right) \star K_{yQ}\,.$$

Consider $h \neq 0$ and take $h \to 0$. For small h, the functions $Y_\pm^{(\alpha)}$ have expansion

$$Y_\pm^{(\alpha)} = 1 + hA_\pm^{(\alpha)} + \cdots\,.$$

The last term behaves as $\log h$, and we get

$$-\log Y_Q = -2\log h \star K_{yQ} + \text{finite terms}\,.$$

Ground state energy: any L, small h

$$-\log Y_Q = -2 \log h \star K_{yQ} + \text{finite terms}\,.$$

Taking into account that $1 \star K_{yQ} = 1$, we conclude

$$Y_Q = h^2 B_Q + \cdots\,,$$

and the ground state energy expands as

$$E_h(L) = -h^2 \int \frac{du}{2\pi} \sum_{Q=1}^{\infty} \frac{d\widetilde{p}^Q}{du} B_Q + \cdots\,.$$

Expanding all the Y-functions around the naive solution up to quadratic order in h

$$Y_Q \approx h^2 B_Q\,, \qquad Y_{\pm}^{(\alpha)} \approx 1 + hA_{\pm}^{(\alpha)} + h^2 B_{\pm}^{(\alpha)}\,,$$

$$Y_{M|vw}^{(\alpha)} \approx A_M^{(\alpha)} + hB_{M|vw}^{(\alpha)} + h^2 C_{M|vw}^{(\alpha)}\,, \qquad Y_{M|w}^{(\alpha)} \approx A_M^{(\alpha)} + hB_{M|w}^{(\alpha)} + h^2 C_{M|w}^{(\alpha)}\,.$$

one can derive equations for the coefficients A's and B's.

Up to the quadratic order the expansion in h is consistent with the conditions

$$B_{M|w}^{(\alpha)} = B_{M|vw}^{(\alpha)} \quad \Leftrightarrow \quad A_{-}^{(\alpha)} = A_{+}^{(\alpha)} = 0$$

Ground state energy: any L, small h

TBA eqs for Q-particles, and w-strings decouple from the eqs for $B^{(\alpha)}_{M|w}$ and $B^{(\alpha)}_{\pm}$

$$-\log B_Q = L\,\widetilde{\mathcal{E}}_Q - \log(1+\frac{1}{A^{(\alpha)}_M})\star K^{MQ}_{vwx}\,, \quad \log A^{(\alpha)}_M = I_{MN}\log(1+A^{(\alpha)}_N)\star s$$

If $A^{(\alpha)}_M$ is constant then since $1\star s = \frac{1}{2}$

$$(A^{(\alpha)}_M)^2 = (1+A^{(\alpha)}_{M-1})(1+A^{(\alpha)}_{M+1}) \quad \Rightarrow \quad A^{(\alpha)}_{M-1} = M^2-1$$

B_Q is computed by using $1\star K^{MQ}_{vwx} = n^{M,Q}_{vwx}$

$$B_Q = 4\,Q^2\,e^{-L\widetilde{\mathcal{E}}_Q}$$

L is quantized!!! if Y_Q is analytic on z-torus.

Thus, the ground state energy at the leading order in h and arbitrary L is given by

$$E_h(L) = -h^2\int\frac{du}{2\pi}\sum_{Q=1}^{\infty}\frac{d\widetilde{p}^Q}{du}\,4\,Q^2\,e^{-L\widetilde{\mathcal{E}}_Q} = -h^2\sum_{Q=1}^{\infty}\int\frac{d\widetilde{p}^Q}{2\pi}\,4\,Q^2\,e^{-L\widetilde{\mathcal{E}}_Q}\,.$$

For $L=2$ the series in Q diverges?! as $\frac{1}{Q}$

Ground state energy: any h, large L

Generalized Lüscher formula

Janik, Lukowski '07

$$E_{\rm gL}(L) = -\int \frac{du}{2\pi} \sum_{Q=1}^{\infty} \frac{d\widetilde{p}^{Q}}{du}\, e^{-L\,\widetilde{\varepsilon}_{Q}}\; {\rm tr}_{Q} e^{i(\pi+h)F} + \cdots .$$

The trace runs through all $16\,Q^2$ polarizations of a Q-particle state. We obtain

$$E_{\rm gL}(L) = -\int \frac{du}{2\pi} \sum_{Q=1}^{\infty} \frac{d\widetilde{p}^{Q}}{du}\, 16\,Q^2 \sin^2\frac{h}{2}\, e^{-L\,\widetilde{\varepsilon}_{Q}} + \cdots .$$

At small values of h it agrees with the previous one.

Expansion of Y-functions in terms of $e^{-L\,\widetilde{\varepsilon}_{Q}}$ is similar to the small h one

$$Y_Q \approx 16\,Q^2 \sin^2\frac{h}{2}\, e^{-L\,\widetilde{\varepsilon}_{Q}}\,, \quad Y_{\pm}^{(\alpha)} \approx 1, \quad Y_{M|w}^{(\alpha)} \approx Y_{M|vw}^{(\alpha)} \approx M(M+2)\,,$$

and the energy of the ground state agrees with the Lüscher formula.

For $h=\pi$ it should give the energy of the non-BPS ground state in the sector with anti-periodic fermions.

Y-system test

It is of interest to compute the contribution Δ Arutyunov, Frolov '09(b)

$$
\begin{aligned}
\Delta &= \log\Big(1-\frac{e^{ih_1}}{Y_-^{(1)}}\Big)\Big(1-\frac{e^{ih_2}}{Y_-^{(2)}}\Big)\big(\theta(-u-2)+\theta(u-2)\big) \\
&+ L\,\check{\mathcal{E}} - \log\Big(1-\frac{e^{ih_1}}{Y_-^{(1)}}\Big)\Big(1-\frac{e^{ih_2}}{Y_-^{(2)}}\Big)\Big(1-\frac{e^{ih_1}}{Y_+^{(1)}}\Big)\Big(1-\frac{e^{ih_2}}{Y_+^{(2)}}\Big)\star\check{K} \\
&- \log\Big(1+\frac{1}{Y_{M|vw}^{(1)}}\Big)\Big(1+\frac{1}{Y_{M|vw}^{(2)}}\Big)\star\check{K}_M + 2\log\big(1+Y_Q\big)\star\check{K}_Q^{\Sigma}\,,
\end{aligned}
$$

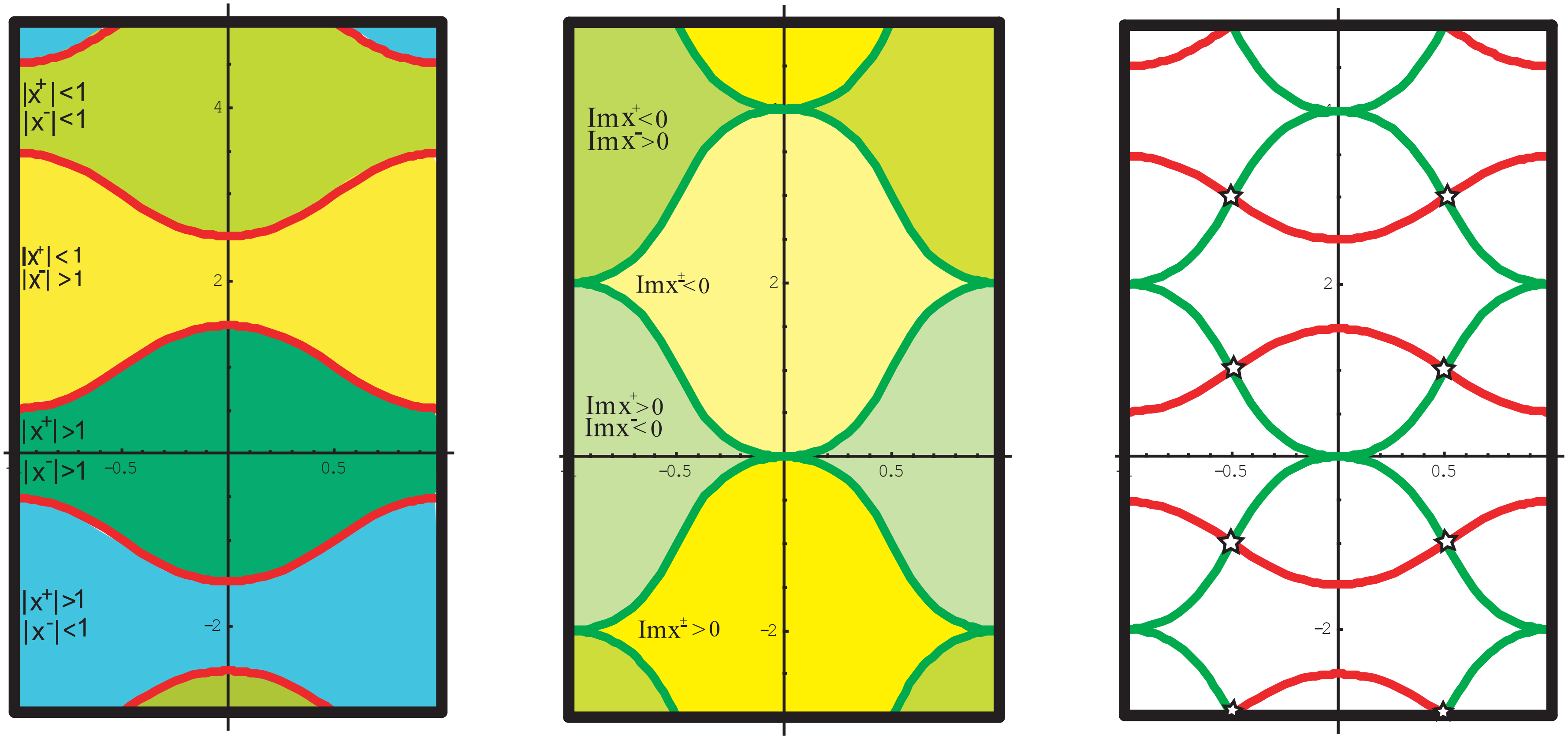
appearing in the simplified set of TBA equations.

TBA eqs may lead to a Y-system only if Δ vanishes on any solution. We get

$$\Delta = L\,\check{\mathcal{E}}\,.$$

- Since Δ does not vanish, the TBA eqs. do NOT lead to an analytic Y-system.
- That means that Y-functions are NOT analytic in the complex u-plane, and have infinitely many cuts.
- This is in contrast to rel. models, and even if the Y-system exists, is it useful?

TBA equations for excited states



TBA equations for excited states

Naive TBA eqs for excited (nonbound) states in the $\mathfrak{su}(2)$ (or $\mathfrak{sl}(2)$?) sector

- Assume that the string theory spectrum is characterized by a set of N real numbers z_k corresponding to momenta of N particles in the large L limit.
- These numbers are determined from the conditions P. Dorey, Tateo '96

$$Y_1(z_k - \frac{\omega_2}{2}) = Y_1(z_{*k}) = -1\,, \quad k = 1, \ldots, N\,,$$

where $Y_1 = e^{-\epsilon_1}$ is the Y-function of the fundamental mirror particles, and it is supposed to be a holomorphic function in a region which contains all z_{*k} and the real mirror momentum line in the z-torus.

- Take the TBA equations for the ground state energy, and deform the integration contour in any integral of the form $f \star K(z) = \int dz' f(z')K(z', z)$ in such a way that all the points z_{*k} lie between the real mirror momentum z line and the integration contour.
- Taking the integration contour back to the real z line, one picks up N extra contributions of the form $-\log S(z_*, z)$ from any term $\log(1 + Y_1) \star K$, where $S(w, z)$ is the S-matrix corresponding to the kernel K:
$K(w, z) = \frac{1}{2\pi i} \frac{d}{dw} \log S(w, z)$.

TBA equations for excited states in the $\mathfrak{su}(2)$-sector

- Q-particles

$$-\ln Y_Q = L\,\widetilde{\mathcal{E}}_Q + \sum_* \log S_{\mathfrak{sl}(2)}^{1_*Q} - \log\left(1+Y_{Q'}\right)\star K_{\mathfrak{sl}(2)}^{Q'Q} - \log\left(1+\frac{1}{Y_{M'|vw}^{(\alpha)}}\right)\star K_{vwx}^{M'Q}$$

$$-\log\left(1-\frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}\right)\star K_-^{yQ} - \log\left(1-\frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}\right)\star K_+^{yQ}\,,$$

- y-particles

$$\ln Y_\pm^{(\alpha)} = \sum_* \log S_\pm^{1_*y} - \log\left(1+Y_Q\right)\star K_\pm^{Qy} + \log\frac{1+\frac{1}{Y_{M|vw}^{(\alpha)}}}{1+\frac{1}{Y_{M|w}^{(\alpha)}}}\star K_M\,.$$

- $M|vw$-strings

$$\ln Y_{M|vw}^{(\alpha)} = \sum_* \log S_{xv}^{1_*M} - \log\left(1+Y_{Q'}\right)\star K_{xv}^{Q'M}$$

$$+\log\left(1+\frac{1}{Y_{M'|vw}^{(\alpha)}}\right)\star K_{M'M} + \log\frac{1-\frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}}{1-\frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}}\star K_M\,.$$

- $M|w$-strings

$$\ln Y_{M|w}^{(\alpha)} = \log\left(1+\frac{1}{Y_{M'|w}^{(\alpha)}}\right)\star K_{M'M} + \log\frac{1-\frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}}{1-\frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}}\star K_M\,.$$

TBA equations for excited states in the $\mathfrak{su}(2)$-sector

- Summation over repeated indices and the index α in the equation for Q-particles is assumed
- The sums in the formulae run over the set of N particles
- All Y-functions depend on the real z (or u) variable of the mirror region
- All integrals are also taken over the real u line or the interval $-2 < u < 2$
- $S^{1_*Q}_{\mathfrak{sl}(2)} \equiv S^{1Q}_{\mathfrak{sl}(2)}(z_*, z)$ is a shorthand notation for the S-matrix with the first and second arguments in the string and mirror regions, respectively
- Finally, both arguments of the kernels in these formulae are in the mirror region

TBA equations for excited states in the $\mathfrak{su}(2)$-sector

Now we take the logarithm of $Y_1(z_{*k}) = -1$, and analytically continue the variable z of $Y_1(z)$ in the TBA eq for Y_1 to the point z_{*k}. This leads to the following exact Bethe equations for the string theory particles momenta p_k

$$\pi i(2n_k + 1) = -\log Y_1(z_{*k}) = -iL\, p_k + \sum_{j=1}^{N} \log S_{\mathfrak{sl}(2)}^{11}(z_{*j}, z_{*k})$$

$$- \log\left(1 + Y_Q\right) \star K_{\mathfrak{sl}(2)}^{Q1} - \log\left(1 + \frac{1}{Y_{M|vw}^{(\alpha)}}\right) \star K_{vwx}^{M1}$$

$$- \log\left(1 - \frac{e^{ih_\alpha}}{Y_-^{(\alpha)}}\right) \star K_-^{y1} - \log\left(1 - \frac{e^{ih_\alpha}}{Y_+^{(\alpha)}}\right) \star K_+^{y1}\,,$$

- $p_k = i\widetilde{\mathcal{E}}_Q(z_{*k})$ is the momentum of the k-th particle
- the second argument in all the kernels in this equation is equal to z_{*k}
- the first argument we integrate with respect to is the original one in the mirror region

TBA equations for excited states in the $\mathfrak{su}(2)$-sector

The energy of the multiparticle state is given by

$$E_{\{n_k\}}(L) = \sum_{k=1}^{N} i\widetilde{p}^{1}(z_{*k}) - \int \mathrm{d}u \sum_{Q=1}^{\infty} \frac{1}{2\pi} \frac{d\widetilde{p}^{Q}}{du} \log\left(1 + Y_Q\right)$$
$$= \sum_{k=1}^{N} \mathcal{E}_k - \int \mathrm{d}u \sum_{Q=1}^{\infty} \frac{1}{2\pi} \frac{d\widetilde{p}^{Q}}{du} \log\left(1 + Y_Q\right) \,,$$

where

$$\mathcal{E}_k = igx_k^- - igx_k^+ - 1 \,,$$

is the energy of a fundamental particle in the string theory.

For practical computations the analytic continuation from the mirror region to the string one is done by introducing the u_*-variable in the string region

$$x_s(u_*) = \frac{u_*}{2}\left(1 + \sqrt{1 - \frac{4}{u_*^2}}\right) \,,$$

with the cut running from -2 to 2. Then, the analytic continuation of all the kernels and S-matrices reduces to the substitution $x^{Q\pm}(u) \to x^{Q\pm}(u_*) \equiv x_s(u_* \pm \frac{i}{g}Q)$.

Conclusions

- The $AdS_5 \times S^5$ string sigma-model can be naturally embedded in the general framework of massive integrable systems.
- Mirror theory is continuum 2-dim quantum field theory, and is closer to usual relativistic models. On the contrary, quantum l.c. string sigma model is rather a lattice theory
- Formulated the *string hypothesis* for the mirror theory — Arutyunov, Frolov '09(a)
- Derived TBA equations for the ground state energy (and excited states) — Arutyunov, Frolov '09(b); Bombardelli, Fioravanti, Tateo '09
- *Different* TBA eqs were proposed by — Gromov, Kazakov, Kozak, Vieira '09

  A different string hypothesis has been apparently used there.
- Simplified the TBA equations
- They lead to the Y-system but only for u in the interval $-2 < u < 2$, and there it agrees with the one conjectured by — Gromov, Kazakov, Vieira '09
- The analyticity of Y_Q on the z-torus implies the quantization of the l.c. momentum or, equivalently, the temperature quantization of the mirror model.

Open problems

- Dressing phase in the mirror theory. Arutyunov,Frolov '09(c)
- Find a proper analytic continuation of the TBA eqs to analyze the excited state energies. The naive continuation does not take into account μ-terms.
- Reproduce known string and field theory results by using the TBA eqs
- Compute numerically anomalous dimension of Konishi for any λ
- Compute analytically anomalous dimension of Konishi up to 12 loops.
- Prove $PSU(2,2|4)$ invariance of the string spectrum
- Prove the gauge independence of the string spectrum