Gauge Invariant Approach to Lagrangian Construction for Higher Spin Fields

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Aim: Brief review of generic approach to problem of Lagrangian formulation for higher spin fields on a gravitational background. The approach is based on BRST - BFV construction and perfectly works both for massless and massive fields, both for bosonic and fermionic fields, and also for the fields with mixed symmetry of indices.

Results and details are published in series of following papers:

I.L.B., A. Pashnev, M. Tsulaia, Phys.Lett., B523 (2001) 338.
X.Bekaert, I.L.B., A. Pashnev, M. Tsulaia, Class.Quant.Grav., 21 (2004) S1457.
I.L.B., V.A. Krykhtin, A. Pashnev, Nucl.Phys., B711 (2005) 367.
I.L.B., V.A. Krykhtin, Nucl.Phys., B727 (2005) 537.
I.L.B., V.A. Krykhtin, L.L. Ryskina, H. Takata, Phys.Lett., B641 (2006) 386.
I.L.B., P.M. Lavrov, V.A. Krykhtin, Nucl.Phys., B762 (2007) 344.
I.L.B., A.V. Galajinsky, V.A. Krykhtin, Nucl.Phys., B779 (2007) 155.
I.L.B., A.V. Galajinsky, JHEP, 0811 (2008) 081.
I.L.B., V.A. Krykhtin, H. Takata, Phys.Lett., B656 (2007) 253.
I.L.B., V.A. Krykhtin, A.A. Reshetnyak, Nucl.Phys., B787 (2007) 211.

I.L.B., A.V. Galajinsky, JHEP, 0811 (2008) 081.
I.L.B., V.A. Krykhtin, L.L. Ryskina, Mod.Phys.Lett., A24 (2009) 401.

PLAN

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• BFV Construction

BFV - construction, which is also called canonical BRST - construction or BRST - BFV - construction was originally developed for covariant operator quantization of gauge theories. As we know, any gauge theory is characterized by first class constraints in phase space T_a, satisfying the following relations in terms of Poisson brackets

$$\{T_a, T_b\} = f^c_{ab} T_c$$

where the structure functions f^c_{ab} can, in principle, depend on phase variables.

Using these constraints one introduces the nilpotent BFV - charge Q by the rule

$$Q = \eta^a T_a + \frac{1}{2} \eta^a \eta^b f^c_{ab} \mathcal{P}_c + \dots$$

$$\{Q, Q\} = 0.$$

Here η^a, $\mathcal{P}_a$ are the canonically conjugate ghost variables, satisfying the relation

$$\{\eta^a, \mathcal{P}_b\} = \delta^a_b.$$

The dots mean the extra terms in ghosts, which should be added to get the nilpotent charge Q in the case when the structure functions f^c_{ab} are not constant.

After quantization, the BFV charge Q becomes a nilpotent Hermitian operator acting in enlarged space of states $|\Psi\rangle$ depending on ghosts. The equation $Q|\Psi\rangle = 0$ defines a physical subspace. It is proved that in physical subspace a unitary S - matrix exists. It is easy to see that the physical states are defined up to transformation

$$|phys'\rangle = |phys\rangle + Q|\Lambda\rangle,$$

which generates the gauge transformations.

It is evident that the BFV - construction in its literal form can not be applied to higher spin field theory for the following reasons:

- BFV - construction is based on a given classical Lagrangian formulation. However just the classical lagrangian formulation is a problem of higher spin field theory.

- BFV - construction is a quantization method. However a problem of Lagrangian formulation for higher spin fields is a problem of classical theory.

• General Scheme of Implementations of BFV Construction in Higher Spin Field Theory

Since a classical higher spin field Lagrangian is a problem, canonical formulation is absent, a system of first class constraints is unknown and the BFV charge can not be constructed by standard way. The only what we can know are the conditions on the fields, which determine the irreducible representations of Poincare or AdS group with given mass and spin or some deformation of these conditions for arbitrary curved space. Existence of such condition on arbitrary curved background is not obvious.

General scheme of implementations of BFV - construction to higher spin fields is based on the following ideas:

- Realization of conditions on fields as operators acting in auxiliary Fock space and treatment of these operator, as first class constraints in terms of (anti)commutators of some unknown yet Lagrangian theory.

- If an algebra of these initial operators is not closed, we add some new operators and demand a complete algebra should be closed. More over, taking in to account construction of Hermitian BFV charge, we must use some operators which can not be treated as the operator constraints. It is worth pointing out that in case of AdS space, the algebra under consideration is non - linear and has the schematic form

$$[T, T] \sim T + T^2.$$

- Construction of BFV charge on the base of above algebra. In case of non - linear algebra, a form of nilpotent BFV charge is not obvious from the very beginning.

- Extending the Fock space by vectors depending on ghost variables and postulating the equation

$$Q|\Phi\rangle = 0$$

which is treated as higher spin field equation of motion.

- Proof that the equation $Q|\Phi\rangle = 0$ reproduces the conditions on the fields we begin with. For example, the relations which determine the irreducible representations of the Poincare or AdS groups are the consequences of the equation $Q|\Phi\rangle = 0$. Since the equation $Q|\Phi\rangle = 0$ is invariant under transformations $|\Phi'\rangle = |\Phi\rangle + Q|\Lambda\rangle$ we automatically obtain gauge invariant equations of motion.

- Construction of Lagrangian leading to equation $Q|\Phi\rangle = 0$. For bosonic fields such a lagrangian has a simple form $\mathcal{L} \sim \langle\Phi|KQ|\Phi\rangle$, where $\langle\Phi_1|\Phi_2\rangle$ is an inner product in extended Fock space and K is some operator depending only or ghost variables. As a result, the lagrangian is written in terms of BFV - charge.

Such a formulation if it is exists, should possess the following features:

- Gauge invariance even in case of massive fields.
- True set of auxiliary fields.
- Unconstrained basic fields.

Conditions of tracelessness or double tracelessness or γ- tracelessness are not imposed from the very beginning, they should be consequences of the equation of motion.

Why we expect that the Lagrangian written in terms of BFV charge should describe a dynamics. The matter is that this operator is constructed on the base of constraints, one of which is on-shell equation of motion.

As a result, the BFV - approach to higher spin field theory is inverse in some sense, to BFV - approach to quantization.

Realization of this scheme faces many specific problems. For example, when we try to construct Hermitian BFV charge, we should add to the set of constraints, the operators which can not be considered as the constraints. If means that the BFV - construction does not work literally and should be generalized. It is interesting to point out that construction of closed algebra in arbitrary curved space imposes a restriction on the space to be a constant curvature space.

• Scheme of Lagrangian Construction in Concrete Higher Spin Field Models

a. **Bosonic massive totally symmetric field in flat space** (I.L.B., V.A. Krykhtin, A. Pashnev, Nucl.Phys., B711 (2005) 367).

Conditions on field $\varphi_{\mu_1\ldots\mu_s}(x)$ determining irreducible representation of Poincare group with mass m and spin s have the form

$$(\partial^2+m^2)\varphi_{\mu_1\ldots\mu_s}=0,$$

$$\partial^{\mu_1}\varphi_{\mu_1\ldots\mu_s}=0,$$

$$\eta^{\mu_1\mu_2}\varphi_{\mu_1\mu_2\mu_3\ldots\mu_s}=0,$$

$$\varphi_{\mu_1\ldots\mu_s}=\varphi_{(\mu_1\ldots\mu_s)}.$$

Consider the Fock space of the vectors

$$|\varphi\rangle=\varphi_{\mu_1\ldots\mu_s}a^{+\mu_1}\ldots a^{+\mu_s}|0\rangle,$$

$$a_\mu|0\rangle=0,$$

where a_μ, $a^{+\mu}$ are the operators satisfying the commutation relations

$$[a_\mu,a_\nu^+]=-\eta_{\mu\nu},$$

$$\eta_{\mu\nu}=diag(+,-,-,\ldots,-).$$

Let us define the operators

$$L_0=-p^2+m^2,$$

$$L_1=a^\mu p_\mu,$$

$$L_2=\frac{1}{2}a^\mu a_\mu,$$

$$p_\mu=-i\partial_\mu.$$

Then one can show that the relations

$$L_0|\varphi\rangle = 0, \qquad L_1|\varphi\rangle = 0, \qquad L_2|\varphi\rangle$$

reproduce the conditions on the fields determining the irreducible representation of the Poincare group. We treat the operators L_0, L_1, L_2 as the constraints in some unknown Lagrangian theory.

To construct an Hermitian BFV charge we need a system of constraints which is invariant under Hermitian conjugation. But the operators L_1 and L_2 are not Hermitian. Therefore we introduce the operators $\hat{L}_1^+ = p_\mu a^{+\mu}$ and $\hat{L}_2^+ = \frac{1}{2}a^{+\mu}a_\mu^+$ and consider and algebra generated by L_0, L_1, L_2, L_1^+, L_2^+. The operators L_1^+ and L_2^+ can not be treated as the constraints on $|\varphi\rangle$ since they annihilate the bra - vector $\langle\varphi|$. It is easy to show that the algebra of above operators is not closed because in process of commutations one gets the new operators m^2 and $G_0 = -a^{+\mu}a_\mu^+ + \frac{D}{2}$. If we add these operators, the algebra is closed.

If to forget for a moment that L_1^+, L_2^+, m^2, G_0 are not the constraints and construct the BFV charge with help of operators L_0, L_1, L_2, L_1^+, L_2^+, m^2, G_0 and consider the equation $Q|\Phi\rangle = 0$ we get $m^2|\varphi\rangle = 0$, where $|\varphi\rangle = |\varphi\rangle + ghost$ dependent terms. If means that such a construction works only in massless case.

To apply the BFV construction in massive case we use a special procedure which is called conversion. Let T is a set of all operators, given in terms of operators $a-\mu$,a_μ^+. We introduce two points of new bosonic oscillators b_1, b_1^+, b_2, b_2^+ and define the operators

$$T_{new} = T + T_{additional},$$

where $T_{additional}$ are constructed in terms of b_1, b_1^+, b_2, b_2^+ such a way that the new operators T_{new} form the same algebra as the operators T but $m^2_{new} = 0$. The corresponding operators $T_{additional}$ exist and can be found. Unfortunately, if to use the usual Hermitian conjugation rules relatively standard Fock space inner product for new oscillators $(b_1)^+ = b_1^+$, $(b_2)^+ = b_2$ then $L_{2additional} \neq (L_{2additional})^+$. To avoid thus problem we change a definition of inner product in sector of new oscillators $\langle\Phi_1|\Phi_2\rangle_{new} = \langle\Phi_1|K|\Phi_2\rangle$ with some operator K. We demand $KL^+_{2additional} = (L^+_{2additional})K$ and the some for all other operator. The corresponding operator K exists and can be found.

Then we construct the extended Fock space of the vectors $|\Phi\rangle = |\varphi\rangle + ghost$ and b - oscillator dependent terms.

The coefficients at ghost and b - dependent terms are treated as auxiliary fields.

Then one can prove:

- The equation $Q|\Phi\rangle = 0$ reproduces the relations on $\varphi_{\mu_1 \ldots \mu_s}$ determining the irreducible representation of the Poincare group.

- The Lagrangian is constructed in terms of vector $|\Phi\rangle$ and operators K and Q.

b. **Bosonic totally symmetric field in arbitrary curved space.**

Let us the to generalize the above construction for arbitrary curved space. First of all one introduces the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + w_\mu^{ab} a_a^+ a_b,$$

where a, b are tangent space indices and w_μ^{ab} a spinor connection. Then we generalize the flat conditions for the field $\varphi_{\mu_1 \ldots \mu_s}(x)$ in the form

$$L_0 = D^2 - m^2 + X,$$
$$L_1 = -i a^\mu D_\mu,$$
$$L_2 = \frac{1}{2} a^\mu a_\mu,$$

with some unknown X.

We will search a most general form of X under the conditions:

- Operator X is Hermitian
- If does not contain the terms with the inverse powers of m
- Acting X on vector $|\varphi\rangle$ does not change rank of tensor $\varphi_{\mu_1 \ldots \mu_s}$.

Then the operator X can be found in explicit form up to some number of numerical coefficients.

We demand the algebra of the operator L_0, L_1, L_2, L_1^+, L_2^+, m^2, G_0 should be closed like in flat case. The commutators among these operators are expressed in terms of curvature tensor and its derivatives. Demanding closure of the algebra imposes the conditions on curvature and one can prove that these conditions for $s > 1$ are valid only in constant curvature space flat, that is in flat space, dS space and AdS space.

c. **Bosonic totally symmetric field in AdS space** (I.L.B., A. Pashnev, M. Tsulaia, Phys.Lett., B523 (2001) 338; I.L.B., P.M. Lavrov, V.A. Krykhtin, Nucl.Phys., B762 (2007) 344).

In this case the operator X can be found in explicit form. Calculation of commutator shows that the algebra of the operators L_0, L_1, L_2, L_1^+, L_2^+, m^2, G_0 is quadratic

$$[T, T] \sim T + T^2.$$

As in flat space for massive case we should use a conversion with help of additional operators $T_{additional}$ and introduce the new inner product.

Construction of nilpotent BFV - charge for non - linear algebra is a very non - trivial problem (see e.g. I.L.B., P.M. Lavrov, J. Math.Phys., 48 (2007) 082306). General solution is absent, the only what we know as existence theorem. Nevertheless, the corresponding BFV can be found in the case under consideration and contains the ghost dependent terms up to 5-th order. Scheme works perfectly and we derive gauge invariant Lagrangian for massive higher spin bosonic totally symmetric field with all appropriate auxiliary and Stukelberg fileds.

d. **Bosonic field with mixed symmetry of indices in flat space** (I.L.B., V.A. Krykhtin, H. Takata, Phys.Lett., B656 (2007) 253).

We consider the field $\varphi_{\mu_1 \dots \mu_{s1} \nu_1 \dots \nu_{s2}}$ corresponding to the following Young tableau

μ_1	μ_2	$\cdots$	$\cdots$	$\cdots$	μ_{s_1}
ν_1	ν_2	$\cdots$	ν_{s_2}		

The conditions on field have the form

$$(\partial^2 + m^2)\varphi_{\mu_1 \dots \mu_{s_1} \nu_1 \dots \nu_{s2}} = 0,$$

$$\partial^{\mu_1}\varphi_{\mu_1 \dots \mu_{s1} \nu_1 \dots \nu_{s2}} = 0,$$

$$\partial^{\nu_1}\varphi_{\mu_1 \dots \mu_{s1} \nu_1 \dots \nu_{s2}} = 0,$$

$$\eta^{\mu_1 \mu_2}\varphi_{\mu_1 \mu_2 \dots \mu_{s1} \nu_1 \dots \nu_{s2}} = 0,$$

$$\eta^{\nu_1 \nu_2}\varphi_{\mu_1 \mu_2 \dots \mu_{s1} \nu_1 \dots \nu_{s2}} = 0,$$

$$\eta^{\mu_1 \nu_1}\varphi_{\mu_1 \mu_2 \dots \mu_{s1} \nu_1 \dots \nu_{s2}} = 0.$$

We introduce two sets of oscillators a_i^μ, $a_i^{+\mu}$, $i = 1, 2$ satisfying the relation

$$[a_i^\mu, a_i^{+\nu}] = -\eta^{\mu\nu}\delta_{ij}$$

and Fock space of the vectors

$$|\varphi\rangle = \varphi_{\mu_1 \dots \mu_{s_1} \nu_1 \dots \nu_{s_2}} a_1^{+\mu_1} \dots a_1^{+\mu_{s_1}} a_2^{+\nu_1} \dots a_2^{+\nu_{s_2}} |0\rangle$$

Also we define the operators

$$L_0 = -p^2 + m^2,$$

$$L_i = a_i^\mu p_\mu,$$

$$L_{ij} = \frac{1}{2} a_i^\mu a_{j\mu},$$

$$G_{12} = -a_1^{+\mu} a_{2\mu}.$$

Then one can prove that the constraints

$$L_0|\varphi\rangle = 0,$$

$$L_i|\varphi\rangle = 0,$$

$$L_{ij}|\varphi\rangle = 0,$$

$$G_{12}|\varphi\rangle = 0$$

are equivalent to above conditions for $\varphi_{\mu_1\dots\mu_{s_1}\nu_1\dots\nu_{s_2}}$.

To close the algebra we add the operators

$$l_i^+ = a_{i\mu}^+ p_\mu,$$

$$l_{ij}^+ = \frac{1}{2} a_i^{+\mu} a_{j\mu}^+,$$

$$g_{12}^+ = g_{21},$$

$$m^2,$$

$$g_{11} = -a_1^{+\mu} a_{1\mu} + \frac{d}{2},$$

$$g_{22} = -a_2^+ a_{2\mu} + \frac{d}{2}.$$

After that the scheme is developed analogously to the case of totally symmetric case.

$\underline{\text{e}}$. **Totally antisymmetric bosonic field in curved space** (I.L.B., V.A. Krykhtin, L.L. Ryskina, Mod.Phys.Lett., A24 (2009) 401).

The totally antisymmetric field $\varphi_{\mu_1 \ldots \mu_p} = \varphi_{(\mu_1 \ldots \mu_p)}$ is not a higher spin field. However the BFV - approach successfully works in this can as well.

We introduce the vector

$$|\varphi\rangle = \varphi_{\mu_1 \ldots \mu_p} a^{+\mu_1} \ldots a^{+\mu_s} |0\rangle,$$

where a_μ, a^+_μ are the fermionic oscillators and the conditions

$$(\nabla^2 - m^2)\varphi + (curvature\ \ dependent\ \ terms) = 0,$$

$$\nabla^{\mu_1} \varphi_{\mu_1 \ldots \mu_p} = 0.$$

The curvature dependent terms are not assumed to be given from the very beginning.

We introduce the operators

$$L_0 = \mathcal{D}^2 - m^2 + X,$$

$$L_1 = -i a^\mu \mathcal{D}_\mu.$$

Operator X is fixed from requirement for the algebra (in terms of anticommutators) to be closed

$$X = R_{\mu\nu\alpha\beta} a^{+\mu} a^\nu a^{+\alpha} a^\beta.$$

There are no any retractions on curvature in this case.

Scheme is perfectly works and we get gauge invariant Lagrangian for massive totally antisymmetric bosonic field. After eliminating all auxiliary and Stuckelberg fields this Lagrangian takes a known form. However our approach leads to various gauge invariant intermediate formulations with auxiliary and Stuckelberg fields.

As to Lagrangian formulation for higher spin fermionic fields, the corresponding aspects will be discussed in the talk by Dr. V.Krykhtin.

Summary and Some Prospects

- Gauge invariant approach to Lagrangian construction for higher spin fields is formulated. The approach perfectly works for massive and massless, bosonic and fermionic fields (for fermionic fields the BFV approach was developd in I.L.B., V.A. Krykhtin, A. Pashnev, Nucl.Phys., B711 (2005) 367; I.L.B., V.A. Krykhtin, L.L. Ryskina, H. Takata, Phys.Lett., B641 (2006) 386; I.L.B., V.A. Krykhtin, A.A. Reshetnyak, Nucl.Phys., B787 (2007) 211) and for fields with mixed symmetry of indices in flat and AdS space. Also this approach perfectly works for totally antisymmetric bosonic and fermionic fields (I.L.B., V.A. Krykhtin, L.L. Ryskina, Nucl.Phys., B819 (2009) 453).

- Lagrangian is formulated in terms of enlarged Fock space and automatically contains all appropriate auxiliary fields. No any off - shell constraints for the fields are imposed (unconstrained formulation, see the discussion in I.L.B., A.V. Galajinsky, V.A. Krykhtin, Nucl.Phys., B779 (2007) 155; I.L.B., A.V. Galajinsky, JHEP, 0811 (2008) 081).

Finally I would like to note that the BRST-BFV approach can be useful for formulating the general methods of derivation of the higher spin field interactions (see first attempts to develop such an approach in I.L., A. Fotopoulos, A.C. Petkou, M. Tsulaia, Phys.Rev., D74 (2006) 105018; A. Fotopoulos, N. Irges, A.C. Petkou, M. Tsulaia, JHEP 0710 (2007) 021). Also, if the Lagrangian is completely formulated in terms of BRST charge, one can hope to derive a procedure for evaluating the quantum effects in higher spin field theories in terms of BRST construction.