A Comparative Study of Laplacians in Riemannian and Antisymplectic Geometry

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May 18, 2009

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Darboux Coordinates

Poisson

Darboux Coordinates

Poisson

Coordinates	q^i	
Momenta	p_i	

Darboux Coordinates

Poisson

Coordinates	q^i	**Boson**	**Fermion**
		$\updownarrow$	$\updownarrow$
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Darboux Coordinates

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Anti-Poisson

Darboux Coordinates

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Anti-Poisson

Fields	ϕ^α
Antifields	ϕ^*_α

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Poisson Bracket $\{\ ,\ \}_{PB}$

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Poisson Bracket $\{\ ,\ \}_{PB}$

$$\begin{aligned} \{q^i, q^j\}_{PB} &= 0 \\ \{q^i, p_j\}_{PB} &= \delta^i_j \\ \{p_i, p_j\}_{PB} &= 0 \end{aligned}$$

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Antibracket $(\ ,\ )_{AB}$

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"Comma is a Boson"

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"Comma is a Fermion"

General Coordinates

General Coordinates

Poisson Bracket

$$\{z^A, z^B\}_{PB} = \omega^{AB}$$

$$\{F, G\}_{PB} = (F\frac{\overleftarrow{\partial^r}}{\partial z^A})\omega^{AB}(\frac{\overrightarrow{\partial^\ell}}{\partial z^B}G)$$

General Coordinates

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Antibracket

$$(z^A, z^B)_{AB} = E^{AB}$$

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Grassmann-parity $\varepsilon(\omega^{AB}) = \varepsilon_A + \varepsilon_B$ **Even**

Antibracket

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Grassmann-parity $\varepsilon(E^{AB}) = \varepsilon_A + \varepsilon_B + 1$ **Odd**

General Coordinates

General Coordinates

Poisson Case

$$\omega^{BA} \;=\; -(-1)^{\varepsilon_A\varepsilon_B}\omega^{AB}$$

- **Antisymmetric**

General Coordinates

Poisson Case

$$\omega^{BA} = -(-1)^{\varepsilon_A \varepsilon_B} \omega^{AB}$$

- Antisymmetric

Inverse 2-form
$\omega = \frac{1}{2} dz^A \omega_{AB} \wedge dz^B$

$$\omega_{BA} = (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} \omega_{AB}$$

- **Skewsymmetric**

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Anti-Poisson Case

$$E^{BA} = -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}E^{AB}$$

- **Antiskewsymmetric**

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Anti-Poisson Case

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- Antiskewsymmetric
- **Morally Symmetric like the Riemannian Case**

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- Antisymmetric
- **Morally Skewsymmetric like the Symplectic Case**

Jacobi Identity and Closeness Relation

Jacobi Identity for Poisson Bracket

$$\sum_{\text{cycl. } f,g,h} (-1)^{\varepsilon_f \varepsilon_h}\{f,\{g,h\}\} \;=\; 0$$

Jacobi Identity and Closeness Relation

Jacobi Identity for Poisson Bracket

$$\sum_{\text{cycl. } f,g,h} (-1)^{\varepsilon_f \varepsilon_h}\{f,\{g,h\}\} \;=\; 0$$

Jacobi Identity for Antibracket

$$\sum_{\text{cycl. } f,g,h} (-1)^{(\varepsilon_f+1)(\varepsilon_h+1)}(f,(g,h)) \;=\; 0$$

Jacobi Identity and Closeness Relation

Jacobi Identity for Poisson Bracket

$$\sum_{\text{cycl. } f,g,h} (-1)^{\varepsilon_f \varepsilon_h}\{f,\{g,h\}\} = 0$$

Jacobi Identity for Antibracket

$$\sum_{\text{cycl. } f,g,h} (-1)^{(\varepsilon_f+1)(\varepsilon_h+1)}(f,(g,h)) = 0$$

Symplectic Case

Closed 2-form:

$$d\omega = 0$$

Jacobi Identity and Closeness Relation

Jacobi Identity for Poisson Bracket

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Symplectic Case

Closed 2-form:

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Jacobi Identity for Antibracket

$$\sum_{\text{cycl. } f,g,h} (-1)^{(\varepsilon_f+1)(\varepsilon_h+1)}(f,(g,h)) = 0$$

Anti-Symplectic Case

Closed 2-form:

$$dE = 0$$

Jacobi Identity and Closeness Relation

Jacobi Identity for Poisson Bracket

$$\sum_{\text{cycl. } f,g,h} (-1)^{\varepsilon_f \varepsilon_h}\{f,\{g,h\}\} = 0$$

Const. rank $\Rightarrow$ Darboux Thm.

Symplectic Case

Closed 2-form:

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Const. rank $\Rightarrow$ Darboux Thm.

Anti-Symplectic Case

Closed 2-form:

$$dE \;=\; 0$$

Riemannian Case

No Jacobi Identity, no Closeness Relation, and no Darboux Thm.

Laplacian

Even Laplacian in Riemannian Case

$$\Delta_\rho = \frac{(-1)^{\varepsilon_A}}{\rho} \frac{\overrightarrow{\partial^\ell}}{\partial z^A} \rho g^{AB} \frac{\overrightarrow{\partial^\ell}}{\partial z^B}$$

Laplacian

Even Laplacian in Riemannian Case

$$\Delta_\rho \;=\; \frac{(-1)^{\varepsilon_A}}{\rho}\frac{\overrightarrow{\partial^\ell}}{\partial z^A}\rho g^{AB}\frac{\overrightarrow{\partial^\ell}}{\partial z^B}$$

Odd Laplacian in Anti-Poisson Case

$$2\Delta_\rho \;=\; \frac{(-1)^{\varepsilon_A}}{\rho}\frac{\overrightarrow{\partial^\ell}}{\partial z^A}\rho E^{AB}\frac{\overrightarrow{\partial^\ell}}{\partial z^B}$$

Laplacian

Even Laplacian in Riemannian Case

$$\Delta_\rho = \frac{(-1)^{\varepsilon_A}}{\rho} \frac{\overrightarrow{\partial^\ell}}{\partial z^A} \rho g^{AB} \frac{\overrightarrow{\partial^\ell}}{\partial z^B}$$

- **Canonical density**

$$\rho_g := \sqrt{g} := \sqrt{\mathrm{sdet}(g_{AB})}$$

Odd Laplacian in Anti-Poisson Case

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- **No canonical density!**

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- Canonical density $\rho_g \;:=\; \sqrt{g} \;:=\; \sqrt{\mathrm{sdet}(g_{AB})}$
- **Δ_ρ^2 is a 4th-order operator.**

Odd Laplacian in Anti-Poisson Case

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- No canonical density!
- **Δ_ρ^2 is a 1st-order operator.**

Laplacian

Even Laplacian in Riemannian Case

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- No canonical density!
- Δ_ρ^2 is a 1st-order operator.
- **When is $\Delta_\rho^2 \;=\; 0$ nilpotent?**

Laplacian

Even Laplacian in Riemannian Case

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- No canonical density!
- Δ_ρ^2 is a 1st-order operator.
- When is $\Delta_\rho^2 = 0$ nilpotent?

The $2 \times 2 = 4$ Classical Geometries and their Symmetries

	Even Geometry	Odd Geometry
Riemannian Covariant Metric	$g = dz^A g_{AB} \vee dz^B$ $\varepsilon(g_{AB}) = \varepsilon_A + \varepsilon_B$ $g_{BA} = -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} g_{AB}$ Antiskewsymmetric No Closeness Relation	$g = dz^A g_{AB} \vee dz^B$ $\varepsilon(g_{AB}) = \varepsilon_A + \varepsilon_B + 1$ $g_{BA} = (-1)^{\varepsilon_A \varepsilon_B} g_{AB}$ Symmetric No Closeness Relation
Inverse Riemannian Contravariant Metric	$\varepsilon(g^{AB}) = \varepsilon_A + \varepsilon_B$ $g^{BA} = (-1)^{\varepsilon_A \varepsilon_B} g^{AB}$ Symmetric Even Laplacian	$\varepsilon(g^{AB}) = \varepsilon_A + \varepsilon_B + 1$ $g^{BA} = (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} g^{AB}$ Skewsymmetric No Laplacian
Symplectic Covariant Two–Form	$\omega = \frac{1}{2} dz^A \omega_{AB} \wedge dz^B$ $\varepsilon(\omega_{AB}) = \varepsilon_A + \varepsilon_B$ $\omega_{BA} = (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} \omega_{AB}$ Skewsymmetric Closeness Relation	$E = \frac{1}{2} dz^A E_{AB} \wedge dz^B$ $\varepsilon(E_{AB}) = \varepsilon_A + \varepsilon_B + 1$ $E_{BA} = -(-1)^{\varepsilon_A \varepsilon_B} E_{AB}$ Antisymmetric Closeness Relation
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Inverse **Symplectic** Contravariant Tensor	$\varepsilon(\omega^{AB}) = \varepsilon_A + \varepsilon_B$ $\omega^{BA} = -(-1)^{\varepsilon_A \varepsilon_B} \omega^{AB}$ Antisymmetric No Laplacian	$\varepsilon(E^{AB}) = \varepsilon_A + \varepsilon_B + 1$ $E^{BA} = -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} E^{AB}$ Symmetric Odd Laplacian

The $2 \times 2 = 4$ Classical Geometries and their Symmetries

	Even Geometry	Odd Geometry
Riemannian Covariant Metric	$g = dz^A g_{AB} \vee dz^B$ $\varepsilon(g_{AB}) = \varepsilon_A + \varepsilon_B$ $g_{BA} = -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} g_{AB}$ Antiskewsymmetric No Closeness Relation	$g = dz^A g_{AB} \vee dz^B$ $\varepsilon(g_{AB}) = \varepsilon_A + \varepsilon_B + 1$ $g_{BA} = (-1)^{\varepsilon_A \varepsilon_B} g_{AB}$ Symmetric No Closeness Relation
Inverse Riemannian Contravariant Metric	$\varepsilon(g^{AB}) = \varepsilon_A + \varepsilon_B$ $g^{BA} = (-1)^{\varepsilon_A \varepsilon_B} g^{AB}$ Symmetric Even Laplacian	$\varepsilon(g^{AB}) = \varepsilon_A + \varepsilon_B + 1$ $g^{BA} = (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} g^{AB}$ Skewsymmetric No Laplacian
Symplectic Covariant Two–Form	$\omega = \frac{1}{2} dz^A \omega_{AB} \wedge dz^B$ $\varepsilon(\omega_{AB}) = \varepsilon_A + \varepsilon_B$ $\omega_{BA} = (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} \omega_{AB}$ Skewsymmetric Closeness Relation	$E = \frac{1}{2} dz^A E_{AB} \wedge dz^B$ $\varepsilon(E_{AB}) = \varepsilon_A + \varepsilon_B + 1$ $E_{BA} = -(-1)^{\varepsilon_A \varepsilon_B} E_{AB}$ Antisymmetric Closeness Relation
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Inverse Riemannian Contravariant Metric	$\varepsilon(g^{AB}) = \varepsilon_A + \varepsilon_B$ $g^{BA} = (-1)^{\varepsilon_A\varepsilon_B} g^{AB}$ Symmetric **Even Laplacian**	$\varepsilon(g^{AB}) = \varepsilon_A + \varepsilon_B + 1$ $g^{BA} = (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} g^{AB}$ Skewsymmetric No Laplacian
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Odd Scalar in Antisymplectic Geometry (KB 2006)

$$\nu_\rho \;:=\; \nu_\rho^{(0)} + \frac{\nu^{(1)}}{8} - \frac{\nu^{(2)}}{24}$$

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What is the most general function $\nu = \nu(z)$ such that

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Unique Answer (up to scaling) (Batalin,KB 2008)

$$\nu = \alpha\, \nu_\rho$$

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$$\Delta \;=\; \Delta_\rho + \nu_\rho$$

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Exponential form

$$\Delta e^{\frac{i}{\hbar}W} = 0$$

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\forall n \geq 3: \quad (M_n,S) &= i(\Delta_\rho M_{n-1}) - \frac{1}{2}\sum_{r=1}^{n-1}(M_r,M_{n-r})
\end{aligned}$$

Khudaverdian's Δ_E Operator

The Δ Operator

$$\Delta \;=\; \Delta_\rho + \nu_\rho$$

$=$ **Odd Laplacian** $+$ **Odd Scalar**
$=$ **built from** E **and** ρ**.**

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Curious Fact

$$\sqrt{\rho}\Delta\frac{1}{\sqrt{\rho}}$$

is independent of ρ!

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Def: Khudaverdian's Δ_E Operator

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Proporties of Δ_E

- Δ_E takes semidensities to semidensities.
- **Δ_E is manifestly independent of ρ.**

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Proporties of Δ_E

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A Comparative Study of Laplacians in Riemannian and Antisymplectic Geometry

The Even Scalar ν_ρ

Even Scalar in Riemannian Geometry with density ρ

$$\nu_\rho \;:=\; \nu_\rho^{(0)} + \frac{\nu^{(1)}}{4} - \frac{\nu^{(2)}}{8} - \frac{\nu^{(3)}}{16}$$

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Terms built from g and ρ

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$$\nu^{(1)} \;:=\; (-1)^{\varepsilon_A}\Big(\frac{\overrightarrow{\partial^\ell}}{\partial z^A} g^{AB} \frac{\overleftarrow{\partial^r}}{\partial z^B}\Big)(-1)^{\varepsilon_B}$$

$$\nu^{(2)} \;:=\; -(-1)^{\varepsilon_C}(z^C,(z^B,z^A))\Big(\frac{\overrightarrow{\partial^\ell}}{\partial z^A} g_{BC}\Big)$$

$$= \; -(-1)^{(\varepsilon_A+1)(\varepsilon_D+1)}\Big(\frac{\overrightarrow{\partial^\ell}}{\partial z^D} g^{AB}\Big) g_{BC}\Big(g^{CD} \frac{\overleftarrow{\partial^r}}{\partial z^A}\Big)$$

$$\nu^{(3)} \;:=\; (-1)^{\varepsilon_A}(g_{AB}, g^{BA})$$ $\leftarrow$ **bracket wrt. metric g.**

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Riemannian Case

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Complete Solution

$$\nu \ = \ \alpha \ \nu_{\rho} + \beta \ \nu_{\rho_g} + \gamma \ (\ln \frac{\rho}{\rho_g}, \ln \frac{\rho}{\rho_g})$$

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$$\Delta_{\rho_g} - \frac{(N-2)R}{(N-1)4} \quad \longrightarrow \quad \Delta_{\rho_g} - \frac{R}{4} \quad \text{for} \quad N \to \infty.$$

Riemannian version Δ_g of Khudaverdian's Δ_E Operator

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$=$ **Even Laplacian** $+$ **Even Scalar**
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$$\sqrt{\rho}\Delta\frac{1}{\sqrt{\rho}}$$

is independent of ρ!

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(starting with DeWitt 1957)

The operator formalism
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A Comparative Study of Laplacians in Riemannian and Antisymplectic Geometry

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Curvature term in Quantum Master Equation

$$(W, W) \ = \ 2i\hbar\Delta_{\rho}W - \hbar^2\frac{R}{4}$$

Important 2-loop effect.

References

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